EFFECT OF BOUNDARY CONDITIONS ON LOCALIZED INSTABILITY OF THE SEMI-INFINITE ORTHOTROPIC PLATE

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ABSTRACT
This paper is concerned with an investigation into the localized instability of a thin elastic orthotropic semi-infinite plate. In this study, a semi-infinite plate, simply supported on two edges and under different boundary conditions of clamped, hinged, sliding contact and free on the other edge, is studied. A mathematical model is used and a general solution is presented. The conditions under which localized solutions exist are investigated.

KEYWORDS: Boundary conditions, Instability, Orthotropic plate

INTRODUCTION
The existence of edge waves along the free edge of a homogeneous and isotropic semi-infinite thin plate, modeled using Kirchhoff theory, was first noted by Konenkov [1]. Konenkov established that, for isotropic plates, precisely one edge wave solution exists for all values of the two free parameters, namely the bending stiffness and Poisson's ratio. The edge wave speed is found to be proportional to and slightly less than the speed of flexural (one-dimensional) waves on a plate of infinite extent. Ambartsumian and Belubekyan [2] considered localized bending waves along the edge of a plate using several non-classical plate theories, concluding that Timoshenko–Mindlin plates do not admit localized edge waves. One of the latest developments in the field has been the localized bending waves in an elastic orthotropic plate, by Mkrtchyan [3].

The analogy between localized vibrations of plates and plate localized instability was established in [4]. Further investigations on the late localized instability problems were also carried out, [5]-[7]. In the present paper, a mathematical model and differential equations are presented, the solutions are found; correspondingly, the necessary and sufficient different conditions for the existence of localized solutions are also investigated. The limiting cases are obtained and finally the results and conclusions are reported.

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MATHEMATICAL MODELING

A semi-infinite plate with two simply supported edges as illustrated in Fig. (1). is considered. The width of the plate is *b* and the thickness is *2h*. The Cartesian coordinate system *(x, y, z)* is chosen so that the plane (*xoy*) is coincident with the plate middle surface, while *z* is the coordinate along the thickness; the *x* axes and *y* are aligned in edges. The plate in Cartesian coordinates is to be defined by the following domain:

$$0 \le x \le \infty \quad ,0 \le y \le b \quad ,-h \le z \le h$$

The plate is uniformly compressed along the edges $y = 0$ and $y = b$ with a constant load *P*. The stability equation for plate middle plane normal displacement $w(x, y)$ can be expressed as [8], [9]:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} + P\frac{\partial^2 w}{\partial y^2} = 0 \quad (1)$$

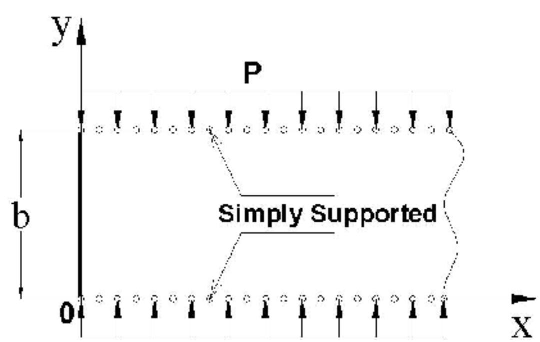

Fig. 1. Uniformly compressed semi-infinite plate simply supported along the edges y=0 and y=b.

where D_{11}, D_{22} are the bending stiffness in the *x, y* direction respectively. Further D_{11}, D_{22}, D_{12} and D_{66} can be written as:

$$D_{11} = \frac{h^3}{12}\frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad D_{22} = \frac{\nu_{21}}{\nu_{12}}D_{11},$$

$$D_{12} = \nu_{12}D_{11}, \quad D_{66} = \frac{h^3}{12}G_{12}$$

and

$$\nu_{21}E_1 = \nu_{12}E_2$$

Here, the subscripts 1 and 2 refer to the *x* and *y* directions, respectively, so E_1 is the Young modulus in the *x* direction, G_{12} is the shear modulus in the *x*-*y* plane, and ν_{12} is the Poisson ratio for transverse strain in the *y* direction caused by stress in the *x* direction, with similar definition for E_2 and ν_{21}.

The boundary conditions on the simply supported edges at y=0, y=b are:

$$\begin{cases} w = 0 \\ \dfrac{\partial^2 w}{\partial y^2} = 0 \end{cases} \qquad y = 0, \quad y = b \quad (2)$$

Different boundary conditions will be considered for the edge x= 0. One additional boundary condition is needed. If the plate is semi-infinite, the localization condition prescribes attenuation as $x \to \infty$, hence an additional constraint is:

$$\lim_{x \to \infty} \; w = 0 \tag{3}$$

General solution of Eq. (1) can be represented as series expansion:

$$w = \sum_{n=1}^{\infty} f_n(x) \sin \lambda_n y, \quad where \quad \lambda_n = n\pi / b \tag{4}$$

Eq. (4) and Eq. (1) yield to the following linear ordinary differential equation and the function $f_n(x)$ can be determined by solving the ordinary differential equation:

$$f_n^{IV} - 2\alpha_1 \lambda_n^2 f_n^{II} + \alpha_2 \lambda_n^4 \left(1 - \eta_n^2\right) f_n = 0 \tag{5}$$

$$\text{where} \quad \alpha_1 = \frac{D_{12} + 2D_{66}}{D_{11}}, \quad \alpha_2 = \frac{D_{22}}{D_{11}}, \quad \eta_n^2 = \frac{P}{D_{22}\lambda_n^2} \tag{6}$$

The attenuation condition of Eq. (3) implies that $f_n(x) \to 0$ as $x \to 0$. Therefore, the general solution of Eq. (5) is in the form:

$$f_n = A_n e^{-p_1 \lambda_n x} + B_n e^{-p_2 \lambda_n x} \tag{7}$$

where p_1 and p_2 are given by:

$$p_{1,2} = \sqrt{\alpha_1 \pm \sqrt{\alpha_1^{\,2} - \alpha_2 \left(1 - \eta_n^{\,2}\right)}} \tag{8}$$

Refering to Eq. (6) it is clear that:

$$\alpha_1 > 0, \quad \alpha_2 > 0 \tag{9}$$

and:

$$\alpha_1 - \sqrt{\alpha_1^{\,2} - \alpha_2 \left(1 - \eta_n^{\,2}\right)} > 0 \quad if \quad 0 \le \eta_n^2 \le 1 \tag{10}$$

The constants A_n and B_n can be obtained imposing the different boundary conditions at edge $x = 0$ leading to a linear homogeneous system in A_n and B_n. The nontrivial solution is given by posing the determinant of the matrix of the coefficients to zero. That yields the equation in η_n.

The different boundary conditions at edge $x = 0$ can be presented as follows:

A. Clamped Edge

The boundary conditions on the clamped edge at $x = 0$ are:

$$w = 0,\ \frac{\partial w}{\partial x} = 0 \qquad at \quad x = 0$$

Substitution of Eq. (4) into above boundary conditions yields:

$$f_n = 0,\ \frac{df_n}{dx} = 0 \qquad at \quad x = 0 \tag{11}$$

Substitution of Eq. (7) into Eq. (11), results in a set of simultaneous equations with regard to A_n and B_n which are obtained as:

$$\left.\begin{array}{l} A_n + B_n = 0 \\ -p_1 A_n - p_2 B_n = 0 \end{array}\right\} \Rightarrow p_2 - p_1 = 0$$

Substitution of Eq. (8) into the above equation results in the following equation:

$$\eta_n{}^2 = 1 + \frac{\alpha_1{}^2}{\alpha_2} > 1 \tag{12}$$

Localized solution doesn't exist, because Eq. (12) doesn't satisfy condition (10).

B. Hinged Edge

The boundary conditions on the hinged edge at $x = 0$ are:

$$w = 0,\ \frac{\partial^2 w}{\partial x^2} = 0 \qquad at \quad x = 0$$

Substitution of Eq. (4) into the above boundary conditions yields:

$$f_n = 0,\ \frac{d^2 f_n}{dx^2} = 0 \qquad at \quad x = 0 \tag{13}$$

By substituting Eq. (7) into Eq. (13), a set of simultaneous equations with regard to A_n and B_n is obtained as follows:

$$\left.\begin{array}{l} A_n + B_n = 0 \\ -p_1{}^2 A_n - p_2{}^2 B_n = 0 \end{array}\right\} \Rightarrow p_2{}^2 - p_1{}^2 = 0 \tag{14}$$

By substituting Eq. (8) into the above equation the following equation is obtained:

$$\eta_n{}^2 = 1 + \frac{\alpha_1{}^2}{\alpha_2} > 1 \tag{15}$$

All localized solution doesn't exist, because Eq. (15) doesn't satisfy condition (10).

C. Sliding Contact

The boundary conditions on the sliding contact edge at $x = 0$ are:

$$\frac{\partial w}{\partial x}=0,\ \ \frac{\partial^3 w}{\partial x^3}=0 \qquad at \quad x=0$$

Substitution of Eq. (4) into above boundary conditions yields:

$$\frac{df_n}{dx}=0,\ \ \frac{d^3 f_n}{dx^3}=0 \qquad at \quad x=0 \tag{16}$$

Substitution of Eq. (7) into Eq. (16), result in a set of simultaneous equations with regard to A_n and B_n which are obtained as:

$$\left.\begin{array}{l} p_1 A_n + p_2 B_n = 0 \\ p_1{}^3 A_n + p_2{}^3 B_n = 0 \end{array}\right\} \Rightarrow p_1 p_2 (\, p_2{}^2 - p_1{}^2\,) = 0 \tag{17}$$

There are two cases:

1. C. $p_2{}^2 - p_1{}^2 = 0$

By substituting Eq. (8) into the above equation the following equation is obtained:

$$\eta_n{}^2 = 1 + \frac{\alpha_1{}^2}{\alpha_2} > 1 \tag{18}$$

All localized solution doesn't exist, because Eq. (18) doesn't satisfy condition (10).

2. C. $p_1 p_2 = 0$

Limiting case (no localization):

$$p_1 p_2 = 0 \Rightarrow p_2 = 0 \Rightarrow \eta_n{}^2 = 1 \quad , \quad p_1 = \sqrt{2\alpha_1} \tag{19}$$

From Eq. (17) and the above equations, $A_n = 0$.

Substituting $A_n = 0$ into (7) the following equation is obtained:
$f_n = B_n$

By substation of the above equation into Eq. (4), the following equation is obtained:

$$w = \sum_{n=1}^{\infty} B_n \, sin\, \lambda_n y \tag{20}$$

The equation Eq. (20) is a lost of stability by cylindrical surface.
From $\eta_n{}^2 = 1$, the minimum of P is obtained as follows:

$$p_n)_{min} = \frac{\pi^2 D_{22}}{b^2} \tag{21}$$

D. Free Edge

The boundary conditions at the free edge $x = 0$ are:

$$\begin{aligned} &M_1 = -(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}) = 0 \\ &\tilde{N}_1 = N_1 + 2\frac{\partial H}{\partial y} = -\frac{\partial}{\partial x}\left[D_{11}\frac{\partial^2 w}{\partial x^2} + (D_{12} + 4D_{66})\frac{\partial^2 w}{\partial y^2}\right] = 0 \end{aligned} \tag{22}$$

where M_1 arises from distribution of in-plane normal stress σ_x and the twisting moment H and shear forces per unit length, N_1 arises from the shear stress in the plate and $\tilde{N}_1$ is reaction force along the edge $x = 0$.

Substitution of Eq. (4) into boundary conditions (22) yields:

$$\left.\begin{aligned} &\frac{d^2 f_n}{dx^2} - \alpha_3 \lambda_n{}^2 f_n = 0 \\ &\frac{d^3 f_n}{dx^3} - (\alpha_3 + 2\alpha_2)\lambda_n{}^2 \frac{df_n}{dx} = 0 \end{aligned}\right. \quad at \quad x = 0 \tag{23}$$

Some new notations are introduced as follows:

$$\alpha_3 = \frac{D_{12}}{D_{11}} \quad , \quad \alpha_4 = \frac{2D_{66}}{D_{11}} \Rightarrow \alpha_1 = \alpha_3 + \alpha_4 \tag{24}$$

where $\alpha_2, \alpha_3, \alpha_4$ are three independent constants.

By using Eq. (6), Eq. (24) and substitution of Eq. (7) into Eq. (23), a set of simultaneous equations with regard to A_n and B_n are obtained as follows:

$$\left.\begin{aligned} &(p_1{}^2 - \alpha_3)A_n + (p_2{}^2 - \alpha_3)B_n = 0 \\ &p_1(p_1{}^2 - \alpha_3 - 2\alpha_4)A_n + p_2(p_2{}^2 - \alpha_3 - 2\alpha_4)B_n = 0 \end{aligned}\right\} \tag{25}$$

The condition that the determinant $\Delta = 0$ yields the characteristic equation as follows:

$$\Delta = p_2(p_1{}^2 - \alpha_3)(p_2{}^2 - \alpha_3 - 2\alpha_4) - p_1(p_2{}^2 - \alpha_3)(p_1{}^2 - \alpha_3 - 2\alpha_4) = 0 \tag{26}$$

Instead of Eq. (26) it is possible to write:

$$(p_2 - p_1)M(\eta) = 0 \tag{27}$$

where

$$M(\eta) = p_1^2 p_2^2 + 2\alpha_4 p_1 p_2 - \alpha_3(p_1^2 + p_2^2) + \alpha_3(\alpha_3 + 2\alpha_4) \tag{28}$$

Using the following equations:

$$p_1^2 + p_2^2 = 2\alpha_1 \quad , \quad \alpha_1 = \alpha_3 + \alpha_4 \tag{29}$$

Eq. (28) can be written as:

$$M(\eta) = p_1^2 p_2^2 + 2\alpha_4 p_1 p_2 - \alpha_3^2 \tag{30}$$

From Eq. (27) there are two cases as follows:

1. D. $p_2 - p_1 = 0$

By the substitution of Eq. (8) into the above equation the following equation is obtained:

$$\eta_n^2 = 1 + \frac{\alpha_1^2}{\alpha_2} > 1 \tag{31}$$

Localized solution doesn't exist, because Eq. (31) doesn't satisfy condition (10).

2. D. $M(\eta) = 0$ (32)

In the first limiting case $\eta_n \to 1 \Rightarrow p_1 = \sqrt{2\alpha_1}, \quad p_2 = 0$

From Eq. (30) the following equation is obtained:

$$M(1) = -\alpha_3^2 < 0 \tag{33}$$

In the second limiting case $\eta_n \to 0 \Rightarrow p_1 p_2 = \sqrt{\alpha_2}$

From Eq. (30) the following equation is obtained:

$$M(0) = \alpha_2 + 2\alpha_4\sqrt{\alpha_2} - \alpha_3^2 \tag{34}$$

$$\alpha_2 + 2\alpha_4\sqrt{\alpha_2} - \alpha_3^2 > 0 \tag{35}$$

Condition (35) is sufficient for the existence of the real root of Eq. (32) in the following interval:

$$0 < \eta_n < 1$$

From Eq. (30) and Eq. (34) the following equation is obtained:

$$p_1 p_2 = -\alpha_4 \pm \sqrt{\alpha_4^2 + \alpha_3^2} \tag{36}$$

From Eq. (8) the following equation is obtained:

$$p_1 p_2 = \sqrt{\alpha_2 (1-\eta_n^2)} \tag{37}$$

From Eq. (36) and Eq. (37) the following equation is obtained:

$$\eta_n^2 = 1-\alpha_2^{-1}(2\alpha_4^2+\alpha_3^2 \pm 2\alpha_4\sqrt{\alpha_4^2+\alpha_3^2}) \tag{38}$$

When condition (35) is not satisfied, there is no root or there are two roots.
For more perspective, two types' of orthotropic materials are considered. Using Eq. (38) and obtaining $\alpha_1, \alpha_2,\ \alpha_3,\ \alpha_4$, the values of η are provided. Then $M(\eta)$ which is monotonous in the interval $0<\eta<1$ for each material is plotted in Fig. (2) and Fig. (3).

1. Plywood

$E_{11} = 1.2x10^5 \quad kg/cm^2$ $\qquad \alpha_1 = 0.18737$

$E_{22} = 0.6x10^5 \quad kg/cm^2$ $\qquad \alpha_2 = 0.50704$

$G_{12} = 0.07x10^5 \quad kg/cm^2$ $\qquad \alpha_3 = 0.071$

$\nu_{12} = 0.071$ $\qquad \alpha_4 = 0.11637$

$\nu_{21} = 0.036$

$\eta_{1,2} = \pm 0.93492$

$\eta_{3,4} = \pm 0.99961$

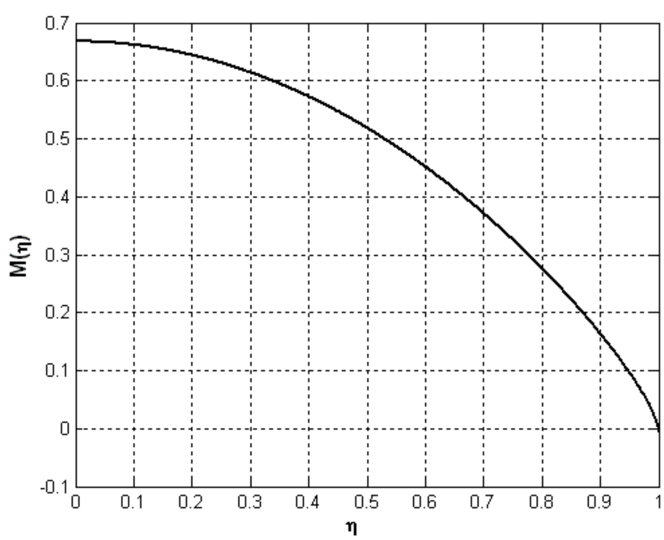

Fig. 2. $M(\eta)$ versus the η of the Plywood material.

2. Carbon/Epoxy Unidirectional Prepreg T300/5208

$E_{11} = 181 \ GPa = 18.46x10^5 \ kg/cm^2$ $\alpha_1 = 0.35885$

$E_{22} = 10.3 \ GPa = 1.05x10^5 \ kg/cm^2$ $\alpha_2 = 0.05689$

$G_{12} = 7.17 \ GPa = 0.731x10^5 \ kg/cm^2$ $\alpha_3 = 0.28$

$\nu_{12} = 0.28$ $\alpha_4 = 0.07884$

$\nu_{21} = 0.01593$

$$\eta_{1,2} = \pm(7.2524x10^{-17} + i1.1846)$$
$$\eta_{3,4} = \pm 0.45792$$

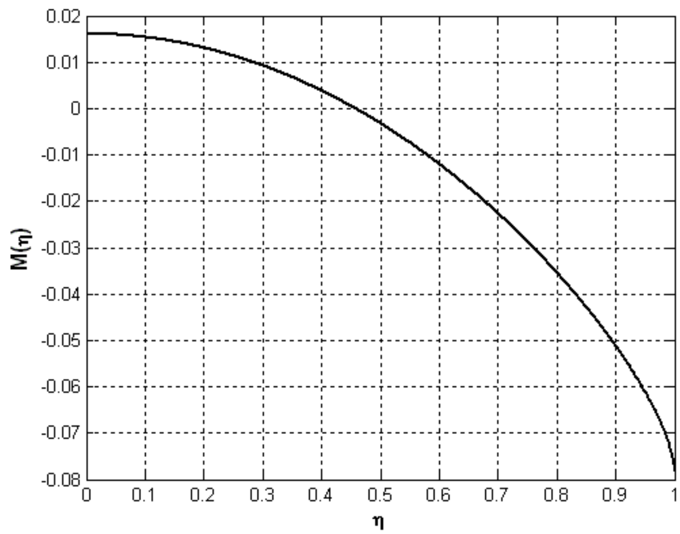

Fig. 3. $M(\eta)$ versus the η of the Carbon/Epoxy T300/5208 material.

CONCLUSIONS

In this paper, localized instability of a thin elastic orthotropic semi-infinite plate has been analyzed. Several conclusions can be summarized as follows:

-In clamped edge conditions localized solution doesn't exist.

-In hinged edge conditions localized solution doesn't exist.

-In sliding contact conditions there are two cases, in one case localized solution doesn't exist and in the other case the equation of loss of stability by cylindrical surface is obtained.

-In free edge there are two cases, in one case localized solution doesn't exist and in the other case real roots are obtained.

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