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On the automorphism group of cubic polyhedral graphs

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Abstract. In the present paper, we introduce the automorphism group of cubic polyhedral graphs whose faces are triangles, quadrangles, pentagons and hexagons.

Keywords: polyhedral graph, automorphism group, fullerene
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1 Introduction

Carbon atoms can bond into very large molecules. Named fullerenes, after U.S. engineer Buckminster Fuller (1895鈥983), these carbon molecules have the same symmetry as a soccer ball, as shown in Figure 1. They are popularly called buckyballs. The most important member of fullerene graphs is C_{60} fullerene with exactly 60 carbon atoms. In general, a fullerene is a cubic planar graph having all faces 5- or 6-cycles, see Figure 2. Examples include the 20-vertex dodecahedral graph, 24-vertex generalized Petersen graph GP(12,2) and graph on 26 vertices truncated icosahedral graph.

A classical fullerene or briefly a fullerene is a cubic three connected graph whose faces entirely composed of pentagons and hexagones and we denote it by a PH-fullerene, see [18, 19]. The non-classical fullerenes are composed of triangles and hexagones or quadrangles and hexagones and we denote them by TH-fullerene or SH-fullerene, respectively. For see

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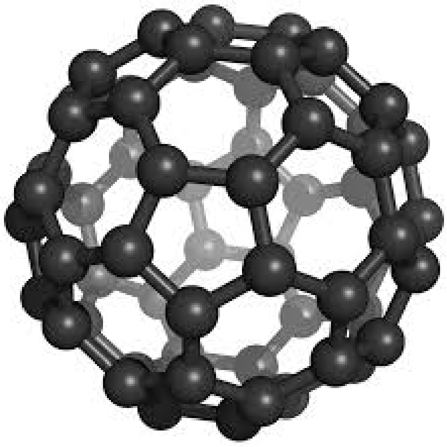

Figure 1. Fullerene C_{60}.

some problems concerning with fullerene graphs and many properties of them are derived, we refer the readers to [1,2,4,7–9] as well as [11,13,16,17,20]. Fullerenes are special cases of a larger class of graphs, namely polyhedral graphs. A polyhedral graph is a three connected simple planar graph and in this paper, we consider only the cubic polyhedral graphs whose faces are a combination of triangles, quadrangles, pentagons and hexagones, see [4,6].

An automorphism of graph $X = (V, E)$ is a bijection β on V which preserves the edge set E. In other words, $e = uv$ is an edge of E if and only if $e^\beta = u^\beta v^\beta$ is an edge of E. Here, the image of vertex u is denoted by u^β. The set of all automorphisms of graph X with the operation of composition is a group on $V(X)$ denoted by $Aut(X)$. Frucht [12] was the first who dealt with graph automorphism. Also quantitative measures based on graph automorphism have been developed, see [3].

Cubic polyhedral graph with t triangular, s quadrilateral, p pentagonal and h hexagonal faces and no other faces is denoted by a $(t,s,p,h)-$polyhedral or briefly a (t,s,p)-polyhedral graph. By these notations, a SPH-polyhedral graph is a planar graph whose faces are quadrangles, pentagons and hexagons. Let m be the number of edges in a given SPH-polyhedral graph F. In [11] Fowler and his co-authors showed that fullerenes are realizable within 28 point groups. In [21] Kutnar et al. proved that for any PH-fullerene graph F, $|Aut(F)|$ divides 120. The present authors in [14] proved that for given TH-fullerene F, $|Aut(F)|$ divides 24 and in [15] they proved that for given SH-polyhedral graph F, $|Aut(F)|$ divides 48. These results are given in the following theorem.

Theorem 1.1. *We have*

- *the size of automorphism group of classical fullerenes divides* 120 *[21].*
- *the size of automorphism group of TSH-fullerenes divides* 24*, [15].*
- *the size of automorphism group of SPH-fullerenes divides* 48*, [21].*

A TPH-polyhedral graph F is one whose faces are triangles, pentagons and hexagons. In this paper, we prove the following theorem.

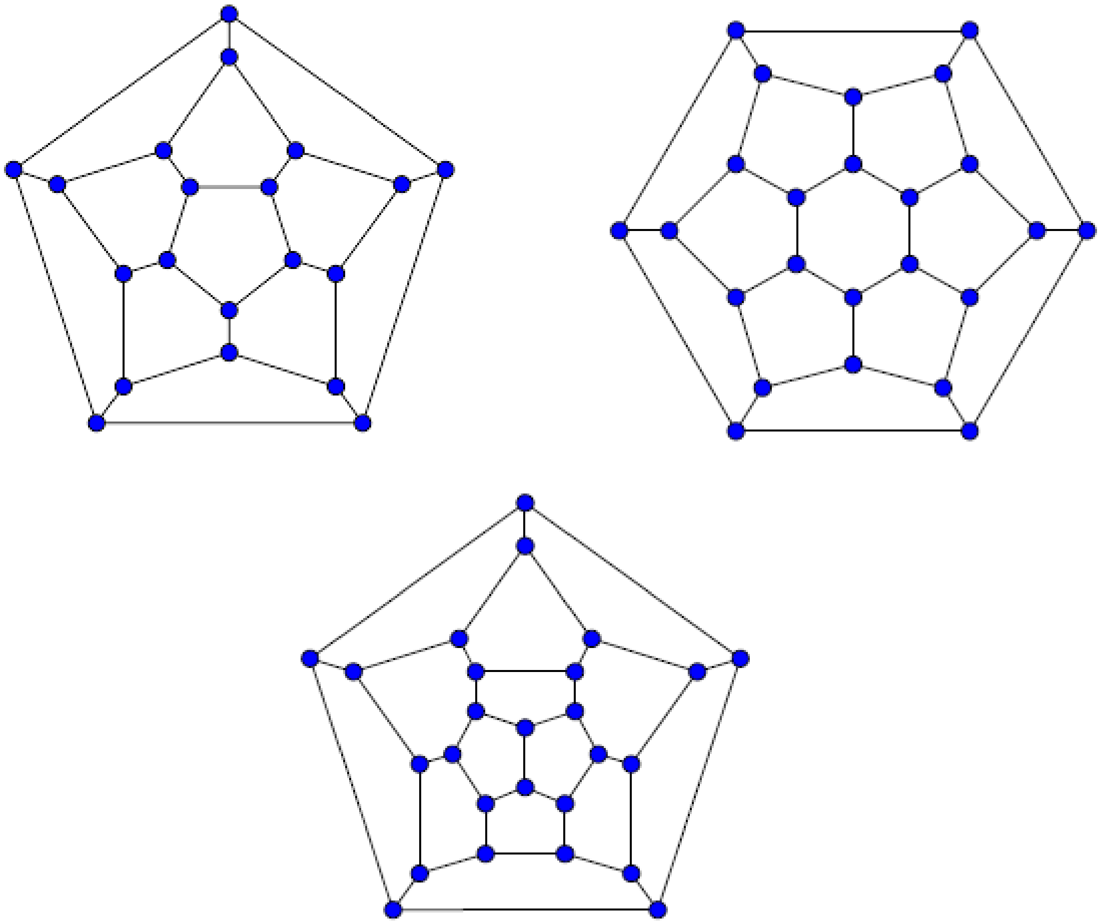

Figure 2. Planar graphs of Fullerenes C_{20},C_{24} and C_{26}.

Theorem A. *Let F be a TPH-polyhedral graph. Then the automorphism group of F is a subgroup of a* $\{2,3,5\}$*-group. Moreover, the order of* $Aut(F)$ *divides* $2^2 \times 3$.

2 Definitions and preliminaries

Let G be a group and Ω a non-empty set. An action of G on Ω denoted by $(G|\Omega)$ induces a group homomorphism φ from G into the symmetric group S_Ω on Ω, where $\varphi(g)^\alpha = g^\alpha$ $(\alpha \in \Omega)$. The orbit of an element $\alpha \in \Omega$ is denoted by α^G and it is defined as the set of all α^g where $g \in G$. Size of Ω is called the degree of this action. The kernel of this action is denoted by $Ker\varphi$. An action is faithful if $Ker\varphi = \{1\}$. The stabilizer of element $\alpha \in \Omega$ is defined as $G_\alpha = \{g \in G | \alpha^g = \alpha\}$. Let $H = G_\alpha$ then for $\alpha, \beta \in \Omega$ $(\alpha \neq \beta)$, H_β is denoted by $G_{\alpha,\beta}$. The orbit-stabilizer theorem implies that $|\alpha^G|.|G_\alpha| = |G|$. For every $g \in G$, let $fix(g) = \{\alpha \in X, \alpha^g = \alpha\}$, then we have:

Lemma 2.1. (Cauchy–Frobenius Lemma) *Let G acts on set Ω, then the number of orbits of G is*

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)|.$$

Example 2.2. Consider the fullerene graph F_{96} depicted in Figure 3. If α denotes the rotation of F_{96} through an angle of 60° around an axis through the midpoints of the front and back faces, then the corresponding permutation is $\alpha = (1,2,3,4,5,6)(7,10,14,17,20,24)(8,11,15,18,21,25)$ $(9,12,16,1923,26)$ $(13,50,58,74,66,42)$ $(22,71,47,39,55,63)$ $(27,28,29,30,31,32)$ $(33,48,56,72,64,40)$ $(34,49,57,73,65,41)(35,51,59,75,67,43)$ $(36,52,60,76,68,44)(37,53,61,77,69,45)$ $(38,54,62,78,70,46)$ $(79,80,81,82,83,84)(85,86,87,88,89,90)$ $(91,96,95,94,93,92)$. Thus, one of orbits of subgroup $\langle\alpha\rangle$ containing the vertex 1 is $1^{\langle\alpha\rangle} = \{1,2,3,4,5,6\}$. Now, consider the axis symmetry element which fixes vertices $\{1,4,8,18,43,44,59,60,85,88, 92,95\}$, the corresponding permutation is $\beta = (2,6)$ $(3,5)$ $(7,9)$ $(10,26)$ $(11,25)(12,24)(13,71)$ $(14,23)$ $(15,21)$ $(16,20)$ $(17,19)(22,50)(27,28)(29,32)(30,31)$ $(33,70)$ $(34,69)$ $(35,67)$ $(36,68)$ $(37,65)$ $(38,64)$ $(39,66)$ $(40,46)$ $(41,45)$ $(42,47)$ $(48,78)(49,77)(51,75)$ $(52,76)$ $(53,73)$ $(54,72)$ $(55,74)(56,62)$ $(57,61)(58,63)(79,80)(81,84)(82,83)(86,90)(87,89)(91,93)(94,96)$.

Let $G = Aut(F_{96})$, clearly $G \geq \langle\alpha,\beta\rangle$ and the orbit-stabilizer property implies that $|G| = |1^G| \times |G_1|$. Any symmetry of the polyhedral graph F_{96} which fixes vertex 1 must also fixes the opposite vertex 4. By applying orbit-stabilizer property, we found that $|G_1| = |4^{G_1}| \times |G_{1,4}|$. It is easy to prove that $|G_{1,4}| = 2$ and hence $|G| = 12$. On the other hand, $|\langle\alpha,\beta\rangle| = 12$, where $\alpha^4 = \beta^2 = 1, \beta^{-1}\alpha\beta = \alpha^{-1}$. This leads us to conclude that $G = \langle\alpha,\beta\rangle \cong D_{12}$.

Example 2.3. Here, we compute the order of automorphism group of polyhedral graph F_{48} depicted in Figure 4. Similar to the last example, if α denotes the rotation of F_{48} through an angle of 90° around an axis through the midpoints of the front and back faces, then the corresponding permutation is $\alpha = (1,3,5,7)$ $(2,4,6,8)(9,15,26,21)$ $(10,16,27,32)$ $(11,17,28,22)(12,29,23)(13,19,30,24)(14,20,31,25)(33,45,41,37)(34,46,42,38)(35,47,43,39)(36,48,44,40)$. Thus $1^{\langle\alpha\rangle} = \{1,3,5,7\}$ and consider the axis symmetry element which fixes no vertices:
$\beta = (1,2)(3,8)(4,7)(5,6)(9,20)(10,19)(11,18)(12,17)(13,16)(14,15)(21,31)(22,29)(23,28)(24,27)(25,26)(30,32)(33,40)(34,39)(35,38)(36,37)(41,48)(42,47)(43,46)(44,45)$.

If $G = Aut(F_{48})$, then $|G| = |2^G| \times |G_2|$ while no element fixes 2. This means that $|G_2| = 1$ and so $|G| = |2^G|$. It is clear that $2^{\langle\alpha,\beta\rangle} = \{1,2,3,4,5,6,7,8\}$ and thus $|2^{\langle\alpha,\beta\rangle}| = 8$. Similar to the last example, we can see that $|\langle\alpha,\beta\rangle| = 8$, where $\alpha^4 = \beta^2 = 1$ and $\beta^{-1}\alpha\beta = \alpha^{-1}$, hence $Aut(F_{48})$ is isomorphic with dihedral group D_8.

3 Main results

Lemma 3.1. *Let F be a TPH-polyhedral graph, with automorphism group $Aut(F)$. Then the stabilizer $Aut(F)_{(u,v,w)}$ of 2-arc (u,v,w) is trivial.*

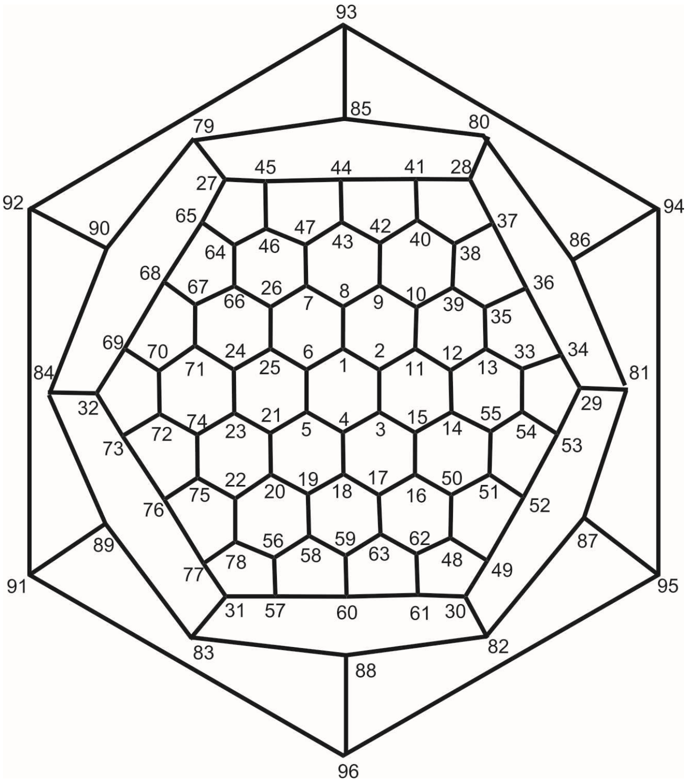

Figure 3. Labeling of fullerene F_{96}.

Proof. It is similar to the proof of [21, Lemma 2]. □

Proposition 3.2. *Let F be a TPH-polyhedral graph with automorphism group $Aut(F)$ and $u \in V(F)$. Then the stabilizer $Aut(F)_u$ of u is trivial or it is isomorphic to one of three groups: the cyclic group $\mathbb{Z}_2$, the cyclic group $\mathbb{Z}_3$ and the symmetric group $\mathbb{S}_3$.*

Proof. It is similar to the proof of [21, Lemma 2]. □

Proof of Theorem A. Let F be a TPH-polyhedral graph with a non-trivial automorphism group, $\mathcal{T}(F)$ be the set of triangles of F and $\mathcal{P}(F)$ be the set of pentagons of F. Let $A = Aut(F)$ and t, p be the number of triangles and pentagons, respectively. We can see that

$$|G| = |K_O| \times |(G/K_O)_T| \times |O| = 2^{\alpha}.3^{\beta}.|O|,$$

and so

$$|G| = 2^{\alpha}.3^{\beta}.5^{\gamma}.|O'|.$$

We distinguish the following cases:

Case 1. $t = 1$ and $p = 9$. We claim that $|Aut(F)|$ divides 3×2. Suppose 2^2 divides $|A|$ and $Syl_2(A)$ is of order 2^2. The order of orbits of $\mathcal{T}(F)$ is 1. By orbit- stabilizer theorem, we have $|K_T| = 2^2$, a contradiction. Let $|Syl_5(A)| = 5$, then $|K_T| = 5$, a contradiction. If $|Syl_3(A)| = 3^2$, then we have $|K_T| = 3^2$, a contradiction.

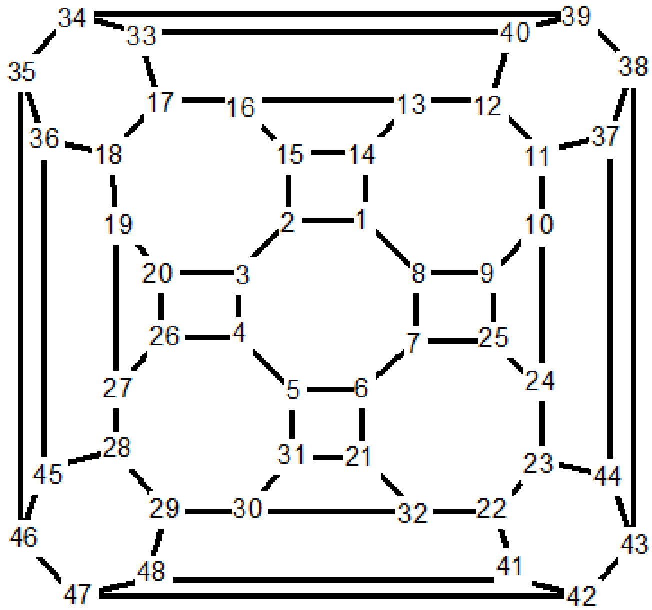

Figure 4. Labeling of fullerene F_{48}.

Case 2. $t = 2$ and $p = 6$, we show that $|Aut(F)|$ divides 3×2^2. Suppose 2^3 divides $|A|$ and $Syl_2(A)$ is of order 2^3. Let $Syl_2(A)$ acts on triangles, then for an orbit O of this action, we have $|O| = 1$ or 2. By orbit-stabilizer theorem, if $|O| = 1$, then $|K_T| = 2^3$, a contradiction and if $|O| = 2$, then $|K_T| = 2^2$, a contradiction. Let $|Syl_5(A)| = 5$, then $|K_T| = 5$, a contradiction. If $|Syl_3(A)| = 3^2$, then we have $|O| = 1$ and so $|K_T| = 3^2$, a contradiction.

Case 3. $t = 3$ and $p = 3$, we prove that $|Aut(F)|$ divides 3×2^2. Let 2^3 divides $|A|$ and $Syl_2(A)$ be of order 2^3 acting on the set of triangles $\mathcal{T}$. Hence, $|O| = 1$ or 2, similar to the last discussion, both of them are contradictions. Also, $|Syl_5(A)| = 5$ is a contradiction. If $|Syl_3(A)| = 3^2$, then the orbits of the action $Syl_3(A)$ on $\mathcal{P}(F)$ are of order 3 and so $|K_P| = 3$, a contradiction.

It should be noted that in a given polyhedral F, no two triangles are adjacent, since F is three connected.

Theorem 3.3. *Let F be a TSH-polyhedral graph. Then $Aut(F)$ is a subgroup of a $\{2,3\}$-group. Moreover, the order of $Aut(F)$ divides* 24.

Proof. By using Euler's theorem, if $s = 0$, then $t = 4$ and then F be TH- fullerene. On the other hand, if $s = 6$, then $t = 0$ and F is a SH-fullerene. Let F is a TSH-polyhedral graph with at least one triangle and one square. We show that $|Aut(F)|$ divides 24. First, we prove that the $Syl_2(F)$ is of order 8. Suppose on the contrary that 2^4 divides $|A|$ and $Syl_2(A)$ is of order 2^4. Let $Syl_2(A)$ acts on the set of triangles, clearly the order of an orbit of an this action is 1 or 2. By orbit-stabilizer theorem, if $|O| = 1$, then $|K_T| = 2^3$, a contradiction and if $|O| = 2$, then $|K_T| = 2^2$, a contradiction. Also $|Syl_5(A)| = 5$ yields a contradiction. If $|Syl_3(A)| = 3^2$, $|K_q| = 3^2$ or $|K_q| = 3$, then we have a contradiction. This completes the proof. □

In [6] the authors derived the list of allowed symmetry groups for each class they constructed the smallest polyhedron for each allowed symmetry. In other words, we have two following theorems, see [5,10,11].

Theorem 3.4. *For the bifaced cubic polyhedra described by the triple (t,s,p), the possible point groups and vertex counts of minimal examples are*

i. $(t,s,p) = (4,0,0)$:
$\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, A_4, S_4$.

ii. $(t,s,p) = (0,6,0)$:
$C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, D_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$
$\mathbb{Z}_2 \times S_3, D_{12}, D_6, \mathbb{Z}_2 \times D_{12}, \mathbb{Z}_2 \times D_6, \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_6$.

iii. $(t,s,p) = (0,0,12)$:
$C_1, \mathbb{Z}_2, A_3,, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, S_6, S_3, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, \mathbb{Z}_2 \times \mathbb{Z}_5, D_{12}, \mathbb{Z}_2 \times S_3,$
$A_4, D_{20}, \mathbb{Z}_2 \ltimes D_{12}, D_{24}, S_4, A_4 \times \mathbb{Z}_2, A_5, \mathbb{Z}_2 \times A_5$.

Theorem 3.5. *For cubic polyhedra with at least two face sizes chosen from $\{3,4,5\}$ and no face of size greater than 6 described by the triple (t,s,p), the possible point groups and vertex counts of minimal examples are*

i. $(t,s,p) = (3,1,1)$:
$C_1, \mathbb{Z}_2$.

ii. $(t,s,p) = (3,0,3)$:
$C_1, \mathbb{Z}_2, A_3, S_3, \mathbb{Z}_2 \times \mathbb{Z}_3$.

iii. $(t,s,p) = (2,3,0)$:
$C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, D_{12}$.

iv. $(t,s,p) = (2,2,2)$:
$C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$.

v. $(t,s,p) = (2,1,4)$:
$C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$.

vi. $(t,s,p) = (2,0,6)$:
$C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, \mathbb{Z}_2 \times S_3, D_{12}$.

vii. $(t,s,p) = (1,4,1)$:
$C_1, \mathbb{Z}_2$.

viii. $(t,s,p) = (1,3,3)$:
$C_1, \mathbb{Z}_2, A_3, S_3$.

ix. $(t,s,p) = (1,2,5)$:
$C_1, \mathbb{Z}_2$.

x. $(t,s,p) = (1,1,7)$:
$C_1, \mathbb{Z}_2$.

xi. $(t,s,p) = (1,0,9)$:
$C_1, \mathbb{Z}_2, A_3, S_3$.

xii. $(t,s,p) = (0,5,2)$:
$C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_5, D_{20}$.

xiii. $(t,s,p) = (0,4,4)$:
$C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, \mathbb{Z}_4$.

xiv. $(t,s,p) = (0,3,6)$:
$C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, A_3, \mathbb{Z}_2 \times \mathbb{Z}_3, S_3, D_{12}$.

xv. $(t,s,p) = (0,2,8)$:
$C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, \mathbb{Z}_2 \times D_8, D_{16}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4$.

xvi. $(t,s,p) = (0,1,10)$:
$C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$.

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