

On the spectrum of Cayley graphs via character table

FARZANEH NOWROOZI AND MODJTABA GHORBANI*

Department of Mathematics, Faculty of Science, Shahid Rajaei

Teacher Training University, Tehran, 16785 – 136, I R. Iran

ABSTRACT. A Cayley graph can be consider as an interpretation of group representation. This means that one can construct the Cayley graph $\Gamma = \text{Cay}(G, S)$ by having the vertex set to be the elements of G with $g \sim h$ if and only if $hg^{-1} \in S$. In this paper, we focus our attention on Cayley graphs with a normal symmetric subset. We also determine all eigenvalues of Buckminster fullerene by constructing $\text{Cay}(G, S)$ where G is the alternating group A_5 .

Keywords: eigenvalue, Cayley graph, characteristic polynomial.

1. INTRODUCTION

An important connection between graph eigenvalues and other branches of mathematics is spectral graph theory. The concepts and methods of spectral graph theory and group theory, bring useful tools to the study of spectrum of Cayley graph. The aim of this paper is to study eigenvalues of Cayley graphs via their character tables. We also apply our method to compute the eigenvalues of fullerenes. Fullerenes are 3-connected cubic graphs composed of 12 pentagons and $n = 0, 2, 3, 4, \dots$ hexagons. The investigation of the mathematical properties of fullerenes are done by mathematicians, see [5, 7, 8, 11, 12, 16-19]. This paper is the first attempt to compute the eigenvalues of fullerene graphs by means of representation theory.

*Author to whom correspondence should be addressed.(e-mail: mghorbani@srttu.edu)

2. DEFINITIONS AND PRELIMINARIES

In this section, we introduce some definitions which will be kept throughout the paper. A simple graph $\Gamma = (V, E)$ is a graph with vertex and edge sets V and E , respectively without loops and multiple edges. When two vertices u and v are endpoints of an edge, we say that they are adjacent and write $u \sim v$ to indicate this. The adjacency matrix A is an $n \times n$ matrix where $n = |V|$ with rows and columns indexed by the elements of the vertex set and the xy -th entry is the number of edges connecting x and y . The characteristic polynomial $P_\Gamma(x)$ of graph Γ with adjacency matrix A is defined as

$$P_\Gamma(x) = \det(xI - A).$$

One can prove that $P_\Gamma(x)$ is a monic polynomial of degree n . The roots of the characteristic polynomial are the eigenvalues of Γ and form the spectrum of Γ . The Cayley-Hamilton theorem says that any linear operator is a zero of its characteristic polynomial.

Theorem 1(Cayley-Hamilton). Let $P_A(x)$ be the characteristic polynomial of A . Then $P_A(A) = 0$.

Since our graphs are undirected, the adjacency matrix A is symmetric. Consequently, all eigenvalues are real. If Γ is a graph on n vertices and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of its adjacency matrix, then the energy and the Estrada index of Γ are defined as [9,10]:

$$E(\Gamma) = \sum_{i=1}^n |\lambda_i| \quad \text{and} \quad EE(\Gamma) = \sum_{i=1}^n e^{\lambda_i}.$$

Let now G is a finite group and S is a generating symmetric subset of G (e.g., $S^{-1} = S$ and $G = \langle S \rangle$) where $1_G \notin S$. The Cayley graph $\Gamma = \text{Cay}(G, S)$ is a connected regular graph whose vertex set is the elements of G and two vertices $g, h \in G$ are adjacent if and only if $hg^{-1} \in S$.

3. MAIN RESULTS AND DISCUSSIONS

The aim of this section is to compute the spectrum of Cayley graphs on finite groups via their character tables. First of all, we recall the concept of character table. For some vector space V , let $GL(V) = \{A \in \text{End}(V) \mid A \text{ is invertible}\}$. A representation of group G is a homomorphism $\rho: G \rightarrow GL(V)$. The dimension of V is called the degree of ρ . A simple example of a representation is the trivial representation $\rho: G \rightarrow \mathbb{C}^*$ given by $\rho(g) = 1$, for all g in G . Let $\varphi: G \rightarrow GL(V)$ be a representation. The character $\chi_\varphi: G \rightarrow \mathbb{C}$ afforded by φ is $\chi_\varphi(g) = \text{tr}(\varphi_g)$. An irreducible character is a character associated with an irreducible representation. The character ρ is called linear if $\chi(1) = 1$. By a classic theorem in representation theory, the number of linear characters of finite group G is $|G/G'|$ where G' denotes the derivative subgroup of G . A character table is a matrix whose rows of it correspond to the irreducible characters whereas the columns

correspond to the conjugacy classes of G . If G is abelian, the eigenvalues of a Cayley graph can easily be determined as follows.

Theorem 2 [6]. Let G be a finite abelian group and S be a symmetric subset of G of size k . Then the eigenvalues of the adjacency matrix of $\text{Cay}(G, S)$ are given by

$$\lambda_\chi = \sum_{s \in S} \chi(s), \quad (1)$$

as χ ranges over all irreducible characters of G .

By using Theorem 2, one can easily see that for the cyclic group $G = \langle x \rangle$ by choosing $S = \{x, x^{-1}\}$, S is symmetric subset of G and $\Gamma \cong \text{Cay}(G, S) \cong C_n$. This implies that

$$\text{spec}(\text{Cay}(\langle x \rangle, \{x, x^{-1}\})) = \text{spec}(C_n),$$

where C_n denotes a cycle graph on n vertices. Let p_i ($1 \leq i \leq n$) be prime numbers and $\alpha_i \in \mathbb{N}$. According to the fundamental abelian theorem, every abelian group G can be presented as follows:

$$G \cong Z_{p_1^{\alpha_1}} \times Z_{p_2^{\alpha_2}} \times \cdots \times Z_{p_k^{\alpha_k}}, \quad (2)$$

where Z_n denotes a cyclic group of order n .

For given graphs Γ_1 and Γ_2 their Cartesian product $\Gamma_1 \square \Gamma_2$ is defined as the graph on the vertex set $V(\Gamma_1) \times V(\Gamma_2)$ with vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ connected by an edge if and only if either $u_1 = v_1$ and $u_2 v_2$ is an edge of $E(\Gamma_2)$ or $u_2 = v_2$ and $u_1 v_1$ in $E(\Gamma_1)$.

Proposition 1[1]. Let $\Gamma_1 = \text{Cay}(G, R)$ and $\Gamma_2 = \text{Cay}(H, T)$ be two Cayley graphs on groups G and H , respectively. Then $\Gamma_1 \square \Gamma_2 = \text{Cay}(G \times H, S)$ is a Cayley graph if and only if

$$S = \{(x, 1), (1, y) \mid x \in G, y \in H\} = (R, 1) \cup (1, T).$$

Proposition 2[15]. Let $\chi_1, \chi_2, \dots, \chi_m$ and $\psi_1, \psi_2, \dots, \psi_n$ be distinct irreducible characters of G and H , respectively. Then the direct product group $G \times H$ has exactly mn irreducible characters $\chi_i \psi_j$ where $1 \leq i \leq m$ and $1 \leq j \leq n$.

By using Propositions 1,2 and Eq. 2, one can compute all eigenvalues of $\text{Cay}(G, S)$ where G is an abelian group. In the following example, we find the spectrum of abelian group $Z_2 \times Z_2$.

Example 1. The cyclic group Z_2 has two conjugacy classes. Consequently, it has two irreducible characters and so its character table is as reported in Table 1.

| CT | 1^2 | 2 |
|-------------|-------|----|
| φ_1 | 1 | 1 |
| φ_2 | 1 | -1 |

Table 1. The character table Z_2 .

Now by using Proposition 2, one can conclude that the group $Z_2 \times Z_2$ has $2 \times 2 = 4$ conjugacy classes and so four irreducible characters. Consider two distinct presentation for Z_2 in which $Z_2 = \{(), (12)\}$ acts on set $\{1, 2\}$ and $Z_2 = \{(), (34)\}$ acts on set $\{3, 4\}$. Let $[()], [(12)]$ and $[()], [(34)]$ be their conjugacy classes, respectively. Then $[()], [(12)], [(34)]$ and $[(12)(34)]$ are conjugacy classes of $Z_2 \times Z_2$. It follows that the character table of $Z_2 \times Z_2$ is as reported in Table 2.

| CT | 1^4 | 21^2 | $1^2 2$ | 2^2 |
|-------------|-------|--------|---------|-------|
| φ_1 | 1 | 1 | 1 | 1 |
| φ_2 | 1 | -1 | 1 | -1 |
| φ_3 | 1 | 1 | -1 | -1 |
| φ_4 | 1 | -1 | -1 | 1 |

Table 2. The character table $Z_2 \times Z_2$.

It is clear that the character table of group $Z_2 \times Z_2$ is the Tensor product of character table of group Z_2 with itself, *e.g.*

$$CT(Z_2 \times Z_2) = CT(Z_2) \otimes CT(Z_2)$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

In general, we have the following proposition.

Proposition 3 [15]. Let G and H be two finite groups with character tables $CT(G)$ and $CT(H)$, respectively. Then the character table of direct product group $G \times H$ is

$$CT(G \times H) = CT(G) \otimes CT(H).$$

In continuing, let G be an arbitrary finite group with a symmetric subset S where $G = \langle S \rangle$. We recall that S is normal subset if and only if $S^g = g^{-1}Sg = S$, for all $g \in G$.

Theorem 3 [6]. Let G be a finite group with a normal symmetric subset S and A be the adjacency matrix of the graph $\Gamma = \text{Cay}(G, S)$. Then the eigenvalues of A are given by

$$\lambda_\chi = \sum_{s \in S} \chi(s) / \chi(1),$$

as χ ranges over all irreducible characters of G . Moreover, the multiplicity of λ_χ is $\chi(1)^2$. Note that in above theorem the λ_χ 's are not distinct.

As an application of above theorem, consider the symmetric group $\text{Sym}(n)$ on n letters. Let $S = \{(i, j) \mid i, j \in \{1, 2, \dots, n\}, i < j\}$, then $\text{Sym}(n) = \langle S \rangle$ and for every $g \in \text{Sym}(n)$, $S^g = S$. This implies that S is a normal symmetric subset of $\text{Sym}(n)$ and by using Theorem 3, all eigenvalues of $\text{Cay}(\text{Sym}(n), S)$ are as follows:

$$\lambda_\chi = |S| \chi((1, 2)) / \chi(1). \quad (3)$$

Since $|S| = n(n-1)/2$, the Eq. 3 can be simplified as follows:

$$\lambda_{\chi} = n(n-1)\chi((1,2))/2\chi(1).$$

Thus, we proved the following result:

Corollary 1. Consider the symmetric group $Sym(n)$ on n letters with a normal symmetric subset $S = \{(i, j) \mid i, j \in \{1, 2, \dots, n\}, i < j\}$. Then an eigenvalue of $Cay(Sym(n), S)$ associated with irreducible character χ is:

$$\lambda_{\chi} = n(n-1)\chi((1,2))/2\chi(1).$$

Example 2. Consider the symmetric group $Sym(5)$ on five letters with a normal symmetric subset $S = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$. The character table of this group is reported in Table 3. Hence, by using Corollary 1, the spectrum of $Sym(5)$ is as follows:

$$[\pm 10]^1, [\pm 5]^{16}, [\pm 2]^{25}, [0]^{36}.$$

The problem of characterizing integral graphs seems to be very difficult and so it is wise to restrict ourselves to certain conditions. Here we are interested to study the symmetric group $Sym(n)$.

| CT | 1 ⁵ | 2 ¹ 3 | 2 ² 1 | 3 ¹ 2 | 3 ² | 4 ¹ | 5 |
|----------|----------------|------------------|------------------|------------------|----------------|----------------|----|
| χ_1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| χ_2 | 1 | -1 | 1 | 1 | -1 | -1 | 1 |
| χ_3 | 4 | 2 | 0 | 1 | -1 | 0 | -1 |
| χ_4 | 4 | -2 | 0 | 1 | 1 | 0 | -1 |
| χ_5 | 5 | -1 | 1 | -1 | -1 | 1 | 0 |
| χ_6 | 5 | 1 | 1 | -1 | 1 | -1 | 0 |
| χ_7 | 6 | 0 | -2 | 0 | 0 | 0 | 1 |

Table 3. The character table of $Sym(5)$.

Theorem 4[15]. Let g be an element of order n in G . Suppose that g is conjugate to g^i for all i with $1 \leq i \leq n$ and $(i, n) = 1$. Then $\chi(g)$ is an integer for all characters χ of G .

Corollary 2. The Cayley graph $\text{Cay}(\text{Sym}(n), S)$ is an integral graph.

Proof. Let $g \in \text{Sym}(n)$ and $(i, o(g))=1$, then the permutations g and g^i have the same cycle type, and hence are conjugate. According to Theorem 4, $\chi(g)$ is an integer for all characters χ of G . Since a rational algebraic integer must be integer, the proof is completed.

4. ENERGY AND ESTRADA OF CAYLEY GRAPHS

The aim of this section is to propose some new results on the energy and Estrada index of Cayley graphs. In [19,20] the authors computed the spectrum of unitary Cayley graphs. Our method for computing the eigenvalues is completely different with their methods. In continuing this paper, the following theorem is crucial.

Theorem 5. Let G and H be two finite groups with normal symmetric subset T_1 and T_2 , respectively. Let $\Gamma_1 = \text{Cay}(G, T_1)$ and $\Gamma_2 = \text{Cay}(H, T_2)$ and $S = (T_1, 1) \cup (1, T_2)$. Then the eigenvalues of $\Gamma_1 \square \Gamma_2 = \text{Cay}(G \times H, S)$ are given by $\lambda_\chi + \lambda_\varphi$ with multiplicity $\chi^2(1)\varphi^2(1)$ as χ, φ range over all irreducible characters of G and H , respectively.

Proof. Let Γ_1 and Γ_2 be two graphs with eigenvalues $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_m , respectively. It is well known fact that $\lambda_i + \mu_j$ ($1 \leq i \leq n, 1 \leq j \leq m$) are eigenvalues of $\Gamma_1 \square \Gamma_2$. Hence, by using Theorem 3, the proof is completed.

Corollary 3. Let S be as defined in Theorem 5 and $\text{Irr}(G)$ denotes the set of irreducible characters of G , then

$$\begin{aligned} E(\text{Cay}(G \times H, S)) &\leq |\text{Irr}(H)| E(G) + |\text{Irr}(G)| E(H), \\ EE(\text{Cay}(G \times H, S)) &= EE(G)EE(H). \end{aligned}$$

Proof.

$$\begin{aligned} E(\text{Cay}(G \times H, S)) &= \sum_{\chi \in \text{Irr}(G), \varphi \in \text{Irr}(H)} |\lambda_{\chi\varphi}| = \sum_{\chi \in \text{Irr}(G), \varphi \in \text{Irr}(H)} |\lambda_\chi + \lambda_\varphi| \\ &\leq \sum_{\chi \in \text{Irr}(G), \varphi \in \text{Irr}(H)} |\lambda_\chi| + |\lambda_\varphi| = \sum_{\chi \in \text{Irr}(G), \varphi \in \text{Irr}(H)} |\lambda_\chi| + \sum_{\chi \in \text{Irr}(G), \varphi \in \text{Irr}(H)} |\lambda_\varphi| \\ &= |\text{Irr}(H)| E(G) + |\text{Irr}(G)| E(H), \\ EE(\text{Cay}(G \times H, S)) &= \sum_{\chi \in \text{Irr}(G), \varphi \in \text{Irr}(H)} e^{\lambda_{\chi\varphi}} = \sum_{\chi \in \text{Irr}(G), \varphi \in \text{Irr}(H)} e^{\lambda_\chi + \lambda_\varphi} = EE(G)EE(H). \end{aligned}$$

In continuing of this section, we determine the energy and Estrada index of fullerene C_{60} by a new method via the concept of Cayley graph. Let A_5 be the alternating group on five symbols. First, we compute the cubic Cayley graph of icosahedral group $I = A_5$. One can prove that the generators of this group are $a =$

$(1,2,3,4,5)$ and $b = (2,3)(4,5)$. By using definition of a Cayley graph, it is easy to see that two elements $x, y \in \Gamma = \text{Cay}(A_5, S)$ are adjacent if and only if $y^{-1}x \in S$. This means that Γ is a cubic graph. Let $S = \{a, a^{-1}, b\}$, then the Cayley graph $\Gamma = \text{Cay}(A_5, S)$ is as depicted in Figure 1.

This Cayley graph is isomorphic with the graph of most stable fullerene namely buckyball fullerene [16,17]. It is well-known fact that among fullerenes just buckyball C_{60} and dodecahedron are Cayley graphs. Further, C_{60} is the only vertex-transitive *IPR* (none of its pentagons make contact with each other) fullerene graph. Recently, the study of graph spectra of such molecules are considered by mathematicians, see for details [2,5,7]. By a direct computation, the spectrum of fullerene C_{60} is determined as follows:

$$3^1, 2.7^3, 2.3^5, 1.8^3, 1.6^4, 1^9, 0.6^5, -0.1^3, -0.4^4, -1.3^4, -1.4^3, -1.6^5, -2^4, -2.57^4, -2.6^3.$$

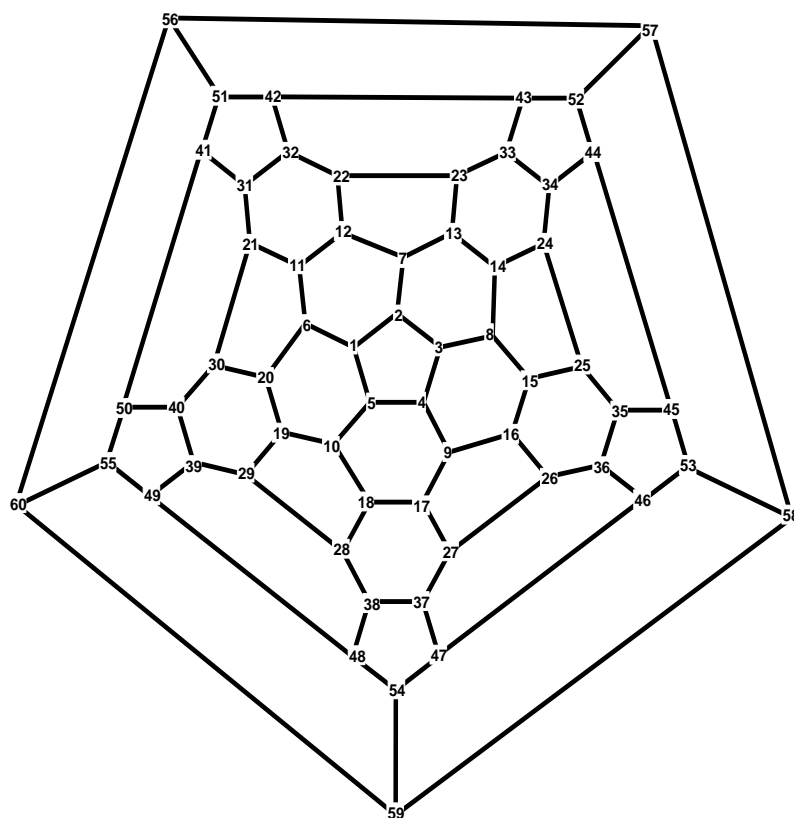


Figure 1. Cayley graph $\Gamma = \text{Cay}(A_5, S)$ where $S = \{(1,2,3,4,5), (1,5,4,3,2), (2,3)(4,5)\}$.

This implies that the energy and Estrada index of C_{60} are approximately 93.16 and 197.45 respectively. Computing the spectrum of a Cayley graph by this method is very difficult. We can apply Theorem 3, to compute the eigenvalues of a Cayley graph. In

other words, for computing the eigenvalues of Cayley graphs by this method, we need the character table of associated group. Hence, in continuing of this section, we compute the eigenvalues of Cayley graphs on groups $I = A_5$ and $I_h = Z_2 \times A_5$, respectively. At first consider the alternating group A_5 . The conjugacy classes of this group are:

$$()^G, (1,2)(3,4)^G, (1,2,3)^G, (1,2,3,4,5)^G, (1,2,3,5,4)^G.$$

Let $S = (1,2,3,4,5)^G \cup (2,3)(4,5)^G$. It is easy to check that $|(1,2,3,4,5)^G| = 12$, $|(2,3)(4,5)^G| = 15$ and so $|S| = 27$. By using Theorem 3, it follows that for each character of I , we have:

$$\lambda_x = \frac{1}{\chi(1)}(|a^G| \chi(a) + |(a^{-1})^G| \chi(b)).$$

Since $|a^G| = |(a^{-1})^G| = 12$ and $|b^G| = 15$, then

$$\lambda_x = \frac{1}{\chi(1)}(24\chi(a) + 15\chi(b)).$$

The character table of A_5 is as follows:

| CT | 1 | b | C | a | a^{-1} |
|----------|---|-----|-----|------|----------|
| χ_1 | 1 | 1 | 1 | 1 | 1 |
| χ_2 | 3 | -1 | 0 | A | $*A$ |
| χ_3 | 3 | -1 | 0 | $*A$ | A |
| χ_4 | 4 | 0 | 1 | -1 | -1 |
| χ_5 | 5 | 1 | -1 | 0 | 0 |

Table 4. Character table of group A_5 .

where $c=(1,2,3)$, $A=-e^{2\pi/5}-e^{8\pi/5}=(1-\sqrt{5})/2$ and $*A=-e^{4\pi/5}-e^{6\pi/5}=(1+\sqrt{5})/2$. The second and fourth columns of Table 4 are corresponded to b and a are $[1,A,*A,-1,0]$ and $[1,-1,-1,0,1]$, respectively. This implies that the spectrum of $\text{Cay}(A_5, S)$ is as follows:

$$\text{spec}(\text{Cay}(I), S) = \begin{pmatrix} -4\sqrt{5}-1 & -6 & 4\sqrt{5}-1 & 3 & 39 \\ 9 & 16 & 9 & 25 & 1 \end{pmatrix}.$$

Thus, the energy and Estrada index of $\text{Cay}(A_5, S)$ are approximately 371 and 0.87×10^{17} .

Consider now group $I_h = Z_2 \times A_5$, one can prove easily that the generators of this group are $\{(1,2,3,4,5), (3,4,5), (6,7)\}$. By using GAP software [20] we deduce that I_h has ten conjugacy classes as follows:

$1 = ()^G, g_2 = (6,7)^G, g_3 = (3,4,5)^G, g_4 = (3,4,5)(6,7)^G, g_5 = (2,3)(4,5)^G, g_6 = (2,3)(4,5)(6,7)^G, g_7 = (1,2,3,4,5)^G, g_8 = (1,2,3,4,5)(6,7)^G, g_9 = (1,2,3,5,4)^G$ and $g_{10} = (1,2,3,5,4)(6,7)^G$. Let a

$= (1,5,3)$ and $b = (2,3)(4,5)(6,7)$. It is easy to see that $T = \{a, a^{-1}, b\}$ is a generating set of I_h and so $S = \{a^G, b^G\}$ is a normal symmetric subset in which $I_h = \langle S \rangle$. By using Theorem 3, it follows that for each character of I_h , the related eigenvalue is:

$$\lambda_\chi = \frac{1}{\chi(1)}(|a^G| \chi(a) + |(a^{-1})^G| \chi(a^{-1}) + |b^G| \chi(b)).$$

Since $|a^G| = 20$ and $|b^G| = 15$, then

$$\lambda_\chi = \frac{1}{\chi(1)}(40\chi(a) + 20\chi(b)).$$

Further, the character table of I_h is as follows:

| CT | 1 | g_2 | g_3 | g_4 | g_5 | g_6 | g_7 | g_8 | g_9 | g_{10} |
|-------------|---|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| χ_1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| χ_2 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 |
| χ_3 | 3 | 3 | 0 | 0 | -1 | -1 | A | A | *A | *A |
| χ_4 | 3 | 3 | 0 | 0 | -1 | -1 | *A | *A | A | A |
| χ_5 | 3 | -3 | 0 | 0 | -1 | 1 | A | -A | *A | -*A |
| χ_6 | 3 | -3 | 0 | 0 | -1 | 1 | *A | -*A | A | -A |
| χ_7 | 4 | 4 | 1 | 1 | 0 | 0 | -1 | -1 | -1 | -1 |
| χ_8 | 4 | -4 | 1 | -1 | 0 | 0 | -1 | 1 | -1 | 1 |
| χ_9 | 5 | 5 | -1 | -1 | 1 | 1 | 0 | 0 | 0 | 0 |
| χ_{10} | 5 | -5 | -1 | 1 | 1 | -1 | 0 | 0 | 0 | 0 |

Table 5. Character table of group $Z_2 \times A_5$.

where $A = -e^{2\pi/5} - e^{8\pi/5} = (1 - \sqrt{5})/2$ and $*A = -e^{4\pi/5} - e^{6\pi/5} = (1 + \sqrt{5})/2$. The third and sixth columns of Table 5 correspond to a and b are $[1, 1, 0, 0, 0, 1, 1, -1, -1]$ and $[1, -1, -1, -1, 1, 1, 0, 0, 1, -1]$, respectively. This implies that the spectrum of $\text{Cay}(I_h, S)$ is

$$\text{spec}(I_h) = \begin{pmatrix} -11 & -5 & 5 & 10 & 25 & 55 \\ 25 & 43 & 18 & 32 & 1 & 1 \end{pmatrix}.$$

Hence, for $\Gamma = \text{Cay}(I_h, S)$, we have $E(\Gamma) \cong 980$ and $EE(\Gamma) \cong 0.77 \times 10^{24}$.

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