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Research Paper

On the Roman domination number of the subdivision of some graphs

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Abstract. A Roman dominating function on a graph $G=(V,E)$ is a function $f:V(G)\rightarrow\{0,1,2\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$. The weight of a Roman dominating function is the value $f(V)=\sum_{u\in V(G)}f(u)$. The minimum possible weight of a Roman dominating function on G is called the Roman domination number of G and is denoted by $\gamma_R(G)$. In this paper, and among some other results, we provide some bounds for the Roman domination number of the subdivision graph $S(G)$ of an arbitrary graph G. Also, we determine the exact value of $\gamma_R(S(G))$ when G is K_n, $K_{r,s}$ or $K_{n_1,n_2,\ldots,n_k}$.

Keywords. Roman domination number, subdivision, bipartite graph, tree.
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1 Introduction and preliminaries

Let $G=(V(G),E(G))$ be a simple, finite and undirected graph. The *open neighborhood* of a vertex $v\in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. For each $S\subseteq V(G)$, $N_G(S)$ is defined as $\cup_{v\in S}N_G(v)$ and the induced subgraph of G on the set S (denoted by $G[S]$) is a subgraph of G with the vertex set S in which two vertices $u,v\in S$ are adjacent just when they are adjacent in G. The *closed neighborhood* of a vertex v in G is $N_G[v]=N_G(v)\cup\{v\}$. The *degree* of a vertex $v\in V(G)$ is $\deg_G(v)=|N_G(v)|$. The *maximum* degree and *minimum* degree are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex is called *universal* if its degree is

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$|V(G)| - 1$. A subset S of $V(G)$ is a *dominating set* for G if each vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G. Recently, the concept of domination is expanded to other parameters of domination such as $k-$rainbow domination and Roman domination. Let $f : V \to \{0,1,2\}$ be an arbitrary function and for each $i \in \{0,1,2\}$, let $V_i = \{v \in V | \ f(v) = i\}$ and $n_i = |V_i|$. These sets determine the function f and hence, one can write $f = (V_0, V_1, V_2)$. Function $f = (V_0, V_1, V_2)$ is a *Roman dominating function*, abbreviated *RDF*, for G if $V_0 \subseteq N_G(V_2)$. If $v \in V_0$ and $u \in N_G(v) \cap V_2$, then we say that u satisfies (or defends) v. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an *RDF* of G, and we say a function $f = (V_0, V_1, V_2)$ to be a γ_R- function if it is an *RDF* for G and $f(V) = \gamma_R(G)$. In recent years much attention drawn to the domination theory which is an interesting branch in graph theory. Cockayne et al. in [7] have shown that for any graph G of order n and maximum degree Δ, $\frac{2n}{\Delta+1} \leq \gamma_R(G)$, and for the classes of paths P_n and cycles C_n, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$. Also they have stated that for the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ with $n_1 \geq n_2 \geq \ldots \geq n_k$ we have

$$\gamma_R(K_{n_1,n_2,\ldots,n_k}) = \begin{cases} n_k + 1 & if \quad 1 \leq n_k \leq 2 \\ 4 & otherwise. \end{cases}$$

Also in [7] it is shown that for each graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ and the lower bound is achieved only when $G = \overline{K_n}$. The Roman domination problem for Johnson graphs is considered in [11]. For further details on Roman domination and its variations we refer the reader to the book chapters [3,4], surveys [5,6] and the article [10].

The subdivision operation of G is an operation in G that replaces any edge by a path and the resulting graph of this operation is called the subdivision graph. If each edge is replaced by a path of length two, then the subdivision graph is denoted by $S(G)$. Note that $V(G) \subseteq V(S(G))$, $|V(S(G))| = |V(G)| + |E(G)|$, $|E(S(G))| = 2|E(G)|$ and each vertex in $V(S(G)) \setminus V(G)$ has degree two. Domination number and identifying code number of the subdivision of some graphs are investigated and determined in [1]. In [9] some bounds for the $2-$rainbow domination number of the subdivision graph $S(G)$ of a graph G is obtained and the exact value of the $2-$rainbow domination number of each tree and some other families of graphs is determined. Some algebraic properties of the subdivision graph of a graph have been studied in [8]. In [2] it is proved that the maximum nullity is equal to the zero forcing number for all complete subdivision graphs.

In this paper, we provide some upper bounds for the Roman domination number of the subdivision of a graph regarding its matchings, connected components and diameter. Also, we provide some tight upper bonds for trees and bipartite graphs. Then we determine the exact value of the Roman domination number of $S(K_n)$, $S(K_{r,s})$ and $S(K_{n_1,n_2,\ldots,n_k})$.

2 Main Results

First of all, we provide some (sharp) bounds for the Roman domination number of the subdivision of an arbitrary graph G. In each one, the bound will be attained when G is equal to its mentioned induced subgraph.

Proposition 2.1. *Let $t \in \mathbb{N}$ be an integer and $H = tK_2$ be an induced subgraph of an n-vertex graph G. Then we have $\gamma_R(S(G)) \leq 2(n-t)$.*

Proof. Define the function $f : V(S(G)) \to \{0,1,2\}$ as

$$f(v) = \begin{cases} 2 & v \in (V(S(H)) \cup V(G)) \setminus V(H) \\ 0 & otherwise. \end{cases}$$

Since H is an induced subgraph of G, we will find out by checking that f is an RDF for $S(G)$ and hence,

$$\gamma_R(S(G)) \leq w(f) = 2t + 2(n-2t) = 2n - 2t.$$ □

Corollary 2.2. *Let G be a graph of order $n \geq 2$. Then $\gamma_R(S(G)) \leq 2n-2$.*

Proof. If $E(G) = \emptyset$, then $S(G) = G$ and the function $f : V(S(G)) \to \{0,1,2\}$ defined by $f(v) = 1$ for each $v \in V(S(G)) = V(G)$, is an RDF and hence

$$\gamma_R(S(G)) = \gamma_R(G) \leq w(f) = n \leq 2n-2.$$

Thus assume that $E(G) \neq \emptyset$ and hence, K_2 is an induced subgraph of G. Now the result follows directly from Proposition 2.1. □

Corollary 2.3. *Let G be a graph of order $n \geq 2$ with s isolated vertices and t connected components of order at least two. Then $\gamma_R(S(G)) \leq 2(n-t) - s$.*

Proof. Choose one edge from each connected component of order at least two to constract an induceed tK_2 in G and consider the function $f : V(S(G)) \to \{0,1,2\}$ as defined in the proof of Proposition 2.1 by modifying it for each isolated vertex $v \in V(G)$ as $f(v) = 1$. Then f is an RDF with weigh $w(f) = 2((n-s)-t) + s = 2(n-t) - s$. □

Proposition 2.4. *If $t \geq 2$ and the path P_t be an induced subgraph in an n-vertex graph G, then $\gamma_R(S(G)) \leq 2(n-t) + \left\lceil \frac{4t-2}{3} \right\rceil$.*

Proof. Since $S(P_t) = P_{2t-1}$ we have $\gamma_R(S(P_t)) = \left\lceil \frac{2(2t-1)}{3} \right\rceil$. Let $f : V(S(P_{2t-1})) \to \{0,1,2\}$ be an optimal RDF for P_{2t-1} and define the function $g : V(S(G)) \to \{0,1,2\}$ as

$$g(v) = \begin{cases} f(v) & v \in V(S(P_{2t-1})) \\ 2 & v \in V(G) \setminus V(P_t) \\ 0 & otherwise. \end{cases}$$

Since P_t is an induced subgraph of G, g is an RDF for $S(G)$ and hence,

$$\gamma_R(S(G)) \leq w(g) = 2(n-t) + \left\lceil \frac{4t-2}{3} \right\rceil.$$ □

Corollary 2.5. *For each n-vertex graph G with diameter d, we have*

$$\gamma_R(S(G)) \leq 2(n-d-1) + \left\lceil \frac{4d+2}{3} \right\rceil.$$

Now in the following result we provide an upper bound for the Roman domination number of the subdivision of an arbitrary bipartite graph G. We will use it for determining the exact value of the Roman domination number of the subdivision of complete bipartite graphs.

Proposition 2.6. *Let G be a bipartite graph with partite sets X and Y in which $|X| \leq |Y|$. Then we have $\gamma_R(S(G)) \leq 2|X| + |Y|$.*

Proof. It is straightforward to check that the function $f : V(S(G)) \to \{0,1,2\}$ defined as

$$f(v) = \begin{cases} 2 & v \in X \\ 1 & v \in Y \\ 0 & otherwise, \end{cases}$$

is an *RDF* for $S(G)$ and hence, $\gamma_R(S(G)) \leq w(f) = 2|X| + |Y|$. □

Since each n-vertex tree is a bipartite graph and the size of its biggest part is at least $\lceil \frac{n}{2} \rceil$, the following result directly follows.

Corollary 2.7. *Let T be an n-vertex tree. Then, $\gamma_R(S(T)) \leq n + \left\lfloor \frac{n}{2} \right\rfloor$.*

In the following result we determine the exact value of the Roman domination number of the subdivision of each complete graph.

Theorem 2.8. *For each $n \geq 2$ we have $\gamma_R(S(K_n)) = 2n - 2$.*

Proof. Since $n \geq 2$ and K_2 is an induced subgraph of K_n, Proposition 2.1 implies that $\gamma_R(S(K_n)) \leq 2n-2$. Let f be an optimal *RDF* for $S(K_n)$ and hence, $w(f) = \gamma_R(S(K_n)) \leq 2n-2$. To complete the proof, it is sufficient to show that $w(f) \geq 2n-2$. For each $i \in \{0,1,2\}$, let $V_i = \{v \in V(K_n) : f(v) = i\}$ and $n_i = |V_i|$. Note that $n = n_2 + n_1 + n_0$. Since f is an *RDF*, for each $v \in V_0$ there exists $z \in N_{S(K_n)}(v)$ such that $f(z) = 2$. Since each $z \in N_{S(K_n)}(V_0)$ with $f(z) = 2$ can satisfy at most two vertices in V_0, we must have

$$\left|\left\{z : z \in N_{S(K_n)}(V_0),\ f(z) = 2\right\}\right| \geq \left\lceil \frac{n_0}{2} \right\rceil.$$

Also, when $u,v \in V_0$ and $N_{S(K_n)}(u) \cap N_{S(K_n)}(v) = \{z\}$, we must have $f(z) \neq 0$. Therefore, if we let

$$\Omega_0 = \left\{z : N_{S(K_n)}(u) \cap N_{S(K_n)}(v) = \{z\} \text{ for some } \{u,v\} \subseteq V_0\right\},$$

then we obtain

$$\sum_{z \in \Omega_0} f(z) \geq \left\lceil \frac{n_0}{2} \right\rceil \times 2 + \left(\binom{n_0}{2} - \left\lceil \frac{n_0}{2} \right\rceil\right) \times 1.$$

If $u, v \in V_1$ and $N_{S(K_n)}(u) \cap N_{S(K_n)}(v) = \{z\}$, then $f(z) \neq 0$. Thus,

$$\sum_{z \in \Omega_1} f(z) \geq \binom{n_1}{2} \times 1$$

in which

$$\Omega_1 = \Big\{ z : \ N_{S(K_n)}(u) \cap N_{S(K_n)}(v) = \{z\} \ for\ some\ \{u, v\} \subseteq V_1 \Big\}.$$

Finally, if $u \in V_0$, $v \in V_1$ and $N_{S(K_n)}(u) \cap N_{S(K_n)}(v) = \{z\}$, then $f(z) \geq 1$. Thus,

$$\sum_{z \in \Omega} f(z) \geq (n_0 n_1) \times 1$$

in which

$$\Omega = \Big\{ z : \ N_{S(K_n)}(u) \cap N_{S(K_n)}(v) = \{z\} \ for\ some\ u \in V_0\ and\ v \in V_1 \Big\}.$$

These facts imply that

$$\begin{aligned} w(f) &= \sum_{v \in V(K_n)} f(v) \quad + \sum_{z \in V(S(K_n)) \setminus V(K_n)} f(z) \\ &\geq n_2 \times 2 + n_1 \times 1 \ + \ \left\lceil \frac{n_0}{2} \right\rceil + \binom{n_0}{2} + \binom{n_1}{2} + n_0 n_1. \qquad (*) \end{aligned}$$

Now we consider the following two cases.

Case 1. $n_0 \neq 0$:

In this case, we have $n_0 \geq 1$ and hence $n_0 n_1 \geq n_1$. Also, $(n_0 - 2)^2 \geq 0$ implies that $n_0 + n_0(n_0 - 1) \geq 4n_0 - 4$ and hence,

$$\frac{n_0}{2} + \frac{n_0(n_0 - 1)}{2} \geq 2n_0 - 2.$$

Thus, from inequality $(*)$ we obtain

$$\begin{aligned} w(f) &\geq 2n_2 + n_1 + n_0 n_1 + \left\lceil \frac{n_0}{2} \right\rceil + \binom{n_0}{2} \\ &\geq 2n_2 + n_1 + n_1 + \frac{n_0}{2} + \frac{n_0(n_0 - 1)}{2} \\ &\geq 2n_2 + 2n_1 + (2n_0 - 2) \\ &\geq 2n - 2, \end{aligned}$$

as desired.

Case 2. $n_0 = 0$:

Since $\left(n_1 - \frac{3}{2}\right)^2 + \frac{7}{4} \geq 0$, we have $\binom{n_1}{2} \geq n_1 - 2$. Now inequality $(*)$ implies that

$$\begin{aligned} w(f) &\geq 2n_2 + n_1 + \binom{n_1}{2} \\ &\geq 2n_2 + n_1 + (n_1 - 2) \\ &= 2n - 2, \end{aligned}$$

and the proof is complete. □

By using Theorem 2.8, we have $\gamma_R(S(K_n)) = 2n - 2$ for each $n \geq 2$, which shows that the bound provided in Corollarly 2.2 is sharp.

Theorem 2.9. *Let $K_{r,s}$ be a complete bipartite graph with $r \leq s$ and $s \geq 2$. Then we have $\gamma_R(S(K_{r,s})) = 2r + s$.*

Proof. By using Proposition 2.6, we have $\gamma_R(S(G)) \leq 2r + s$. Let $f : V(S(K_{r,s})) \to \{0,1,2\}$ be an *RDF* of minimum weight for $S(K_{r,s})$. Thus, $w(f) = \gamma_R(S(K_{r,s})) \leq 2r + s$. In order to complete the proof, it is sufficient to show that $w(f) \geq 2r + s$. Assume that X and Y are two partite sets of $K_{r,s}$ in which $|X| = r$ and $|Y| = s$. For each $i \in \{0,1,2\}$, let

$$X_i = \{x : \; x \in X, \; f(x) = i\}, \; Y_i = \{y : \; y \in Y, \; f(y) = i\}, \; r_i = |X_i|, \; s_i = |Y_i|.$$

Note that $r = r_0 + r_1 + r_2$ and $s = s_0 + s_1 + s_2$. Since f is an *RDF*, for each $v \in X_0 \cup Y_0$ there exists $z \in N_{S(K_{r,s})}(v)$ such that $f(z) = 2$. When $x_0 \in X_0$ and $y_0 \in Y_0$ and

$$\{z\} = N_{S(K_{r,s})}(x_0) \cap N_{S(K_{r,s})}(y_0),$$

we must have $f(z) \geq 1$. Also, if for $x_0 \in X_0$, $y_0 \in Y_0$, $N_{S(K_{r,s})}(x_0) \cap N_{S(K_{r,s})}(y_0) = \{z\}$ we have $f(z) = 2$, then z can satisfy both of x_0 and y_0. Thus,

$$\sum_{z \in \Omega'} f(z) \geq \max\{r_0, s_0\} \times 2 + \left(r_0 s_0 - \max\{r_0, s_0\}\right) \times 1,$$

in which

$$\Omega' = \left\{ z : \; N_{S(K_{r,s})}(x_0) \cap N_{S(K_{r,s})}(y_0) = \{z\} \; \textit{for some} \; x_0 \in X_0 \; \textit{and} \; y_0 \in Y_0 \right\}.$$

If $x_1 \in X_1$, $y_1 \in Y_1$ and z is the unique common neighbor of x_1 and y_1 in $S(K_{r,s})$, then we have $f(z) \geq 1$. If $x_0 \in X_0$, $y_1 \in Y_1$ and z is the unique common neighbor of x_0 and y_1, then $f(z) \geq 1$. A similar argument holds when $x_1 \in X_1$ and $y_0 \in Y_0$. Therefore,

$$\begin{aligned} w(f) &= \sum_{x \in X} f(x) \; + \; \sum_{y \in Y} f(y) \; + \sum_{z \in V(S(K_{r,s})) \setminus V(K_{r,s})} f(z) \\ &\geq (2r_2 + r_1) + (2s_2 + s_1) \; + \; r_0 s_0 + max\{r_0, s_0\} + r_1 s_1 + r_0 s_1 + r_1 s_0 \\ &= (2r_2 + r_1) + (2s_2 + s_1) \; + \; max\{r_0, s_0\} + r_0(s_0 + s_1) + r_1(s_0 + s_1) \\ &= (2r + s) + (s_2 - s_0 - r_1 - 2r_0) + max\{r_0, s_0\} + r_0(s_0 + s_1) + r_1(s_0 + s_1). \quad (**) \end{aligned}$$

Now we consider the following cases.

Case 1. $s_0 + s_1 \geq 2$:
In this case, we have

$$max\{r_0, s_0\} + r_0(s_0 + s_1) + r_1(s_0 + s_1) \geq s_0 + 2r_0 + 2r_1.$$

Now inequality $(**)$ implies that

$$\begin{aligned} w(f) &\geq (2r + s) + (s_2 - s_0 - r_1 - 2r_0) + (s_0 + 2r_0 + 2r_1) \\ &= (2r + s) + s_2 + r_1 \\ &\geq 2r + s, \end{aligned}$$

as desired.

Case 2. $s_0 + s_1 = 1$:
In this case $s_0 \leq 1$ and since $s \geq 2$ we have

$$s_2 = s - (s_0 + s_1) = s - 1 \geq 1 \geq s_0.$$

Thus,

$$max\{r_0, s_0\} + r_0(s_0 + s_1) + r_1(s_0 + s_1) \geq r_0 + r_0 + r_1.$$

Then by using the inequality $(**)$ we obtain

$$\begin{aligned} w(f) &\geq (2r + s) + (s_2 - s_0 - r_1 - 2r_0) + (r_0 + r_0 + r_1) \\ &= (2r + s) + s_2 - s_0 \\ &\geq 2r + s, \end{aligned}$$

as desired.

Case 3. $s_0 + s_1 = 0$:
In this case we have $s_0 = 0$ and $s = s_2$. Hence,

$$max\{r_0, s_0\} + r_0(s_0 + s_1) + r_1(s_0 + s_1) = max\{r_0, s_0\} \geq r_0.$$

Then, inequality $(**)$ implies that

$$\begin{aligned} w(f) &\geq (2r + s) + (s_2 - s_0 - r_1 - 2r_0) + (r_0) \\ &= (2r + s) + s_2 - r_1 - r_0 \\ &= (2r + s) + s - r_1 - r_0 \\ &\geq (2r + s) + r - r_1 - r_0 \\ &= (2r + s) + r_2 \\ &\geq 2r + s. \end{aligned}$$

Now the proof is complete. □

In the following we provide a lemma which will be used in the next theorem.

Lemma 2.10. *Let $k \geq 3$ be an integer and $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph of order $n = n_1 + n_2 + \cdots + n_k$. Then,*

i) if $k = 3$, then $|E(G)| \geq 2n - 3$,

ii) if $k \geq 4$, then $\frac{n}{2} + |E(G)| \geq 2n$.

Proof. Let $X^1, X^2, \ldots, X^k$ be partite sets of $K_{n_1,n_2,\ldots,n_k}$ in which $|X^i| = n_i$, $1 \leq i \leq k$. For each $i \in \{1, 2, \ldots, k\}$ choose an arbitrary vertex $x_1^i \in X^i$. Obviously, the set $\{x_1^1, x_1^2, \ldots, x_1^k\}$ induces a complete subgraph K_k in $K_{n_1,n_2,\ldots,n_k}$. If there exists $x \in X^i \setminus \{x_1^i\}$, then x has at least $k-1$ neighbors in other parts and hence, provides at least $k-1$ edges. Thus,

$$|E(G)| \geq \binom{k}{2} + (n-k)(k-1).$$

If $k = 3$, then we have

$$|E(G)| \geq \binom{3}{2} + (n-3)(3-1) = 2n - 3,$$

which confirms the statement (i). Assume that $k \geq 4$ and hence, $(n - k + \frac{n}{2}) \geq 2$. Now for the statement (ii) we obtain

$$\begin{aligned}
\frac{n}{2} + |E(G)| &\geq \frac{n}{2} + \binom{k}{2} + (n-k)(k-1) \\
&\geq \frac{n}{2} + \binom{k}{2} + (n-k) \times 3 \\
&= 2n + \frac{k(k-1)}{2} - 2k + n - k + \frac{n}{2} \\
&= 2n + \frac{k^2 - 5k}{2} + \left(n - k + \frac{n}{2}\right) \\
&\geq 2n + \frac{k^2 - 5k}{2} + 2 \\
&= 2n + \frac{(k-1)(k-4)}{2} \\
&\geq 2n.
\end{aligned}$$

□

For the Roman domination number of the subdivision of complete multipartite graphs we have the following interesting result.

Theorem 2.11. *Let $k \geq 3$ be an integer and $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph of order $n = n_1 + n_2 + \ldots + n_k$ in which $n_1 \geq n_2 \geq \ldots \geq n_k$ and $n_1 \geq 2$. Then, we have $\gamma_R(S(G)) = 2n - n_1$.*

Proof. Assume that $V(G) = X^1 \cup X^2 \cup ... \cup X^k$ in which $X^i = \{x_1^i, x_2^i, ..., x_{n_i}^i\}$ is the i-th part of G for each $i \in \{1,2,...,k\}$, and $V(S(G)) = V(G) \cup Z$ where

$$Z = \left\{ x_{rs}^{ij} \,:\, i \neq j,\ \{i,j\} \subseteq \{1,2,...,k\},\ 1 \leq r \leq n_i,\ 1 \leq s \leq n_j,\ N_{S(G)}(x_{rs}^{ij}) = \{x_r^i,\ x_s^j\} \right\}.$$

Define the function $f : V(S(G)) \to \{0,1,2\}$ as

$$f(v) = \begin{cases} 1 & v \in X^1 \\ 2 & v \in V(G) \setminus X^1 \\ 0 & v \in Z. \end{cases}$$

With a simple check, we can see that f is an RDF for $S(G)$ and hence,

$$\gamma_R(S(G)) \leq w(f) = 2n - n_1.$$

Now let $g : V(S(G)) \to \{0,1,2\}$ be an RDF for $S(G)$ with the minimum weight. Thus, $w(g) = \gamma_R(S(G)) \leq 2n - n_1$. To complete the proof, we want to show that $w(g) \geq 2n - n_1$. We consider the following two cases.

Case 1. $g(v) \in \{0,2\}$ for each $v \in V(G)$:
If $g(v) = 2$ for each $v \in V(G)$, then we have

$$w(g) \geq \sum_{v \in V(G)} g(v) = 2n,$$

which contradicts the fact $w(g) \leq 2n - n_1$. Hence, there exists $v \in V(G)$ such that $g(v) = 0$. For each $i \in \{1,2,...,k\}$, let $X_0^i = \{v :\ v \in X^i,\ g(v) = 0\}$. Also, let $V_0 = X_0^1 \cup X_0^2 \cup \cdots \cup X_0^k$ and $k' = \left|\{i :\ X_0^i \neq \emptyset\}\right|$. Note that we have $k' \geq 1$.

If $k' = 1$, then $V_0 = X_0^j$ for some $j \in \{1,2,...,k\}$ and since g is an RDF, each vertex $v \in X_0^j$ has at least one (private) neighbor in the set Z which is assigned 2 by the function g. This implies that

$$w(g) = \sum_{v \in V(G)} g(v) + \sum_{z \in Z} g(z) = 2(n - |V_0|) + \sum_{z \in Z} g(z) \geq 2(n - |V_0|) + 2|V_0| = 2n,$$

which contradicts the inequality $w(g) \leq 2n - n_1$.
Therefore, we have $k' \geq 2$ and hence, $|V_0| \geq k' \geq 2$. Thus, the induced subgraph of G on the set V_0 (i.e. $G[V_0]$) is a complete (bipartite or) multipartite graph with k' partite sets. Since g is an RDF, each vertex $v \in V_0$ has at least one neighbor in the set Z which is assigned 2 by the function g, and for each vertex in Z whose both neighbors are in V_0 we must have $g(z) \geq 1$.

Also, each vertex $z \in Z$ with $g(z) = 2$ can satisfy at most two vertices in V_0. Therefore,

$$\begin{aligned} w(g) &= \sum_{v \in V(G)} g(v) + \sum_{z \in Z} g(z) \\ &= 2(n - |V_0|) + \sum_{z \in Z} g(z) \\ &\geq 2(n - |V_0|) + \left\lceil \frac{|V_0|}{2} \right\rceil \times 2 + \left(\left|E(G[V_0])\right| - \left\lceil \frac{|V_0|}{2} \right\rceil \right) \times 1 \\ &= 2(n - |V_0|) + \left\lceil \frac{|V_0|}{2} \right\rceil + \left|E(G[V_0])\right|. \qquad (***) \end{aligned}$$

If $k' \geq 4$, then the statement (ii) in Lemma 2.10 (applied to the complete k'-partite graph $G[V_0]$) and the inequality $(***)$ imply that

$$w(g) \geq 2(n - |V_0|) + 2|V_0| = 2n > 2n - n_1,$$

which is a contradiction.
If $k' = 3$, then we have $|V_0| \geq k' = 3$. Then, the statement (i) in Lemma 2.10 by using the inequality $(***)$ imply that

$$\begin{aligned} w(g) &\geq 2(n - |V_0|) + \left\lceil \frac{|V_0|}{2} \right\rceil + (2|V_0| - 3) \\ &= 2n + \left\lceil \frac{|V_0|}{2} \right\rceil - 3 \\ &\geq 2n + \left\lceil \frac{3}{2} \right\rceil - 3 \\ &= 2n - 1, \end{aligned}$$

which contradicts the fact $w(g) \leq 2n - n_1 \leq 2n - 2$.
Thus, we must have $k' = 2$. This means that $G[V_0]$ is a complete bipartite graph and $V_0 = X_0^i \cup X_0^j$ for some $i, j \in \{1, 2, ..., k\}$. At first, assume that $G[V_0]$ is a star graph (i.e. it is isomorphic to $K_{1,|V_0|-1}$) and hence, $\left|E(G[V_0])\right| = |V_0| - 1$. Without loss of generality, we can assume that $|X_0^i| = 1$ and $|X_0^j| = |V_0| - 1$. Since for each $v \in X_0^j$ we have $g(v) = 0$, each $v \in X_0^j$ has a (private) neighbor $z \in Z$ with $g(z) = 2$. Thus,

$$\begin{aligned} w(g) &= \sum_{v \in V(G)} g(v) + \sum_{z \in Z} g(z) \\ &= 2(n - |V_0|) + \sum_{z \in Z} g(z) \\ &\geq 2(n - |V_0|) + 2(|V_0| - 1) \\ &= 2n - 2 \\ &\geq 2n - n_1, \end{aligned}$$

as desired.
Now assume that $G[V_0]$ is not a star graph. Hence, each of its partite sets contains at least

two vertices and $|V_0| \geq 4$. Hence,

$$\left|E(G[V_0])\right| = |X_0^i|\,|X_0^j| \geq \min\left\{\ q\,(|V_0|-q):\ q \in \{2,3,...,|V_0|-2\}\ \right\} = 2(|V_0|-2).$$

Now inequality $(***)$ implies that

$$\begin{aligned} w(g) &\geq 2(n-|V_0|) + \left\lceil \frac{|V_0|}{2} \right\rceil + \left|E(G[V_0])\right| \\ &\geq 2(n-|V_0|) + \left\lceil \frac{|V_0|}{2} \right\rceil + 2(|V_0|-2) \\ &= 2n + \left\lceil \frac{|V_0|}{2} \right\rceil - 4 \\ &\geq 2n + \left\lceil \frac{4}{2} \right\rceil - 4 \\ &\geq 2n - n_1, \end{aligned}$$

which completes the proof in this case.

Case 2. There exists $v \in V(G)$ such that $g(v) = 1$:

Note that $V(G) = X^1 \cup X^2 \cup \cdots \cup X^k$. Hence, there exists a vertex $x_r^i \in X^i$ with $g(x_r^i) = 1$ for some $i \in \{1,2,...,k\}$ and some $r \in \{1,2,...,n_i\}$. At first, assume that for each $z = x_{rs}^{ij} \in N_{S(G)}(x_r^i)$ we have $g(z) = 0$. Then, since g is an RDF and $N_{S(G)}(x_{rs}^{ij}) = \{x_r^i, x_s^j\}$, we must have $g(x_s^j) = 2$ for each $j \in \{1,2,...,k\} \setminus \{i\}$ and each $s \in \{1,2,...,n_j\}$. Also, since g is an RDF, for each $v \in X^i$ we have

$$\left(g(v) + \sum_{z \in N_{S(G)}(v)} g(z)\right) \geq 1.$$

Therefore, we obtain

$$\begin{aligned} w(g) &= \sum_{v \in V(G)\setminus X^i} g(v) + \sum_{v \in X^i} g(v) + \sum_{z \in Z} g(z) \\ &= (n-n_i) \times 2 + \sum_{v \in X^i} g(v) + \sum_{z \in Z} g(z) \\ &\geq 2(n-n_i) + \sum_{v \in X^i} \left(g(v) + \sum_{z \in N_{S(G)}(v)} g(z)\right) \\ &\geq 2(n-n_i) + n_i \\ &= 2n - n_i \\ &\geq 2n - n_1, \end{aligned}$$

as desired.

Thus, we can suppose that for each vertex $v \in V(G)$ with $g(v) = 1$ there exists $z \in N_{S(G)}(v)$ such that $g(z) \neq 0$. If $g(z) = 2$, then the function $g_1 : V(S(G)) \to \{0,1,2\}$ defined by

$$g_1(u) = \begin{cases} 0 & u = v \\ g(u) & otherwise, \end{cases}$$

is an *RDF* and $w(g_1) < w(g)$ which contradicts the minimality of $w(g)$.
Therefore, if $v \in V(G)$ is a vertex with $g(v) = 1$, then for each $z \in N_{S(G)}(v)$ we have $g(z) \in \{0,1\}$ and there exists $z \in N_{S(G)}(v)$ such that $g(z) = 1$.
Assume that $v \in V(G)$, $g(v) = 1$ and $g(z) = 1$ where $z \in N_{S(G)}(v)$. Define the function $g_2 : V(S(G)) \to \{0,1,2\}$ as

$$g_2(u) = \begin{cases} 0 & u = v \\ 2 & u = z \\ g(u) & otherwise. \end{cases}$$

It is straightforward to see that g_2 is an *RDF* and $w(g_2) = w(g)$. Thus, we can replace g with g_2. Note that by using this algorithm, we reduce the number of vertices in $V(G)$ which are assigned 1. By repeating this algorithm if necessary, we obtain an *RDF* with the minimum weight, say h, such that $h(v) \in \{0,2\}$ for each $v \in V(G)$. Now Case 1 can be applied to complete the proof. □

3 Conclusions

In this paper, we provide some sharp bounds for the Roman domination number of the subdivision of a graph regarding its matchings, connected components and diameter. Then, in each case we show that the bound will be attained when G is equal to a mentioned induced subgraph. Also, we provide some tight upper bonds for bipartite graphs which lead to some upper bounds for trees. Finally, we determine the exact value of the Roman domination number for the subdivision of of complete graph K_n, complete bipartite graph $K_{r,s}$ and complete multipartite graph $K_{n_1,n_2,\dots,n_k}$.

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