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Exploring the watching system of polyhedral graphs

Sheyda Maddah*

Department of Mathematics, Faculty of Science, Shahid Rajaee Teacher Training University, Tehran, 16785-163, I. R. Iran

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Abstract. Watching system in a graph G is a finite set $W = \{w_1, w_2, ..., w_k\}$ where each w_i is a couple $w_i = (v_i, Z_i)$, and where v_i is a vertex and $Z_i \subseteq NG[v_i]$ such that $\{Z_1, ..., Z_k\}$ is an identifying system. The concept of watching system was first introduced by Auger in [1] and this system provide an extension of identifying code in the sense that an identifying code is a particular watching system. In this paper, we determine the watching system of specific Cayley graphs.

Keywords: watching systems, generelized Peteersen graph, identifying codes
Mathematics Subject Classification (2010): 20B25. Secondary: 05C35, 05C38..

1 Introduction

By a graph we mean a collection of points and lines connecting them and we call them vertices and edges, respectively. We also denote them by $V(G)$ and $E(G)$, respectively. Two vertices x and y are adjacent if $xy \in E(G)$. The distance between two arbitrary vertices $x, y \in V(G)$ is defined as the length of the shortest path connecting them denoted by $d(x, y)$. A connected graph is one whose all pairs of vertices are connected by a path. A simple graph is a graph without loop and parallel edges. A molecular graph is a labeled simple graph whose vertices and edges correspond to the atoms and chemical bonds, respectively.

Domination in graph theory is a natural model form any location problems in operations research. It has many other applications in dominating queens problem, school bus routing problem, computer communication network problems, social network theory, land survey-

*Corresponding author(*Emial address*:sheydaaa.1989@gmail.com)
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ing and kernels of games. Domination theory is required for encryption of binary string into a DNA sequence. A dominating set of a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one vertex in D.

Let Ω be a symmetric generating set of a finite group Γ. Assume that (Γ, Ω) be such that $\Gamma = \langle \Omega \rangle$ and Ω satisfies the two conditions

c_1: the identity element $e \notin \Omega$ and

c_2: if $a \in \Omega$, then $a^{-1} \in \Omega$.

Given (Γ, Ω) satisfying c_1 and c_2 define a Cayley graph $G = Cay(\Gamma, \Omega)$ with $V(G) = \Gamma$ and $E(G) = \{(x, y)_a | x, y \in \Gamma, a \in \Omega\ and y = xa\}$.

In 1998, Karpovsky et al. introduced the notion of identifying code, ambitioned to model systems for detection and localization of faults in multiprocessors networks. Identifying codes have been investigated in many directions.

The notion of watching system was introduced by Auger et al. in [1, 2]. These systems provide an extension of identifying codes in the sense that an identifying code is a particular watching system. It can be proved that the watching number is greater than or equal with domination number, see also [3–5].

Let $G = (V(G), E(G))$ be an undirected graph. A watcher w of G is a couple $w = (v_i, z_i)$, where v_i belongs to $V(G)$ and z_i is a subset of vertices set of $N_G(V_i)$. If a vertex v belongs to v_i, we say that v is covered by w. Two vertices v_1 and v_2 in G are said to be separated by a set of watchers if the list of the watchers covering v_1 is different from that of v_2. We say that a set W of watchers is a watching system for G, if every vertex v is covered by at least one $w \in W$, and any two vertices v_1,v_2 are separated by W. The minimum number of watchers necessary to watch G is denoted by $w(G)$. Here, we determine the watching number of Cayley graphs $Cay(Z_2 \times Z_{p^\alpha}, S)$ where $S = (\varphi_2 \times \varphi_{p^\alpha})$ for $p = 1, 2, 3, 5, 7$.

2 Watching System

A watching system in a graph $G = (V(G), E(G))$ is a finite set $W = \{w_1, w_2, ..., w_k\}$ where each w_i is a couple system $\{(v, v) : v \in V(G)\}$. Therefore, W is a watching system of G if every vertex is covered by at least one watcher in W and any two distinct vertices are separated by at least one watcher in W.

The edge $e = uv$ is called watched edge if u or v covered by a watcher. Let us recall that a dominating set in G is a subset D of V(G) such that every vertex not in D is adjacent to at least one element in D. Let respectively $w(G)$, $\gamma(G)$ and $i(G)$ denote the minimum sizes of a watching system, of a dominating set, and when it exists of an identifying code in G. These parameters will be called watching number, domination number, and identifying number, respectively.

Lemma 2.1. *[2] For any graph G, we have*

$$\lceil log_2(|V(G)| + 1) \rceil \leq w(G). \tag{1}$$

Lemma 2.2. *[2] For any graph G, we have*

$$\gamma(G) \leq w(G) \leq \gamma(G)\lceil log_2(\Delta(G)+2)\rceil. \tag{2}$$

Lemma 2.3. *Let G be a graph and H be a spaning sub graph of G, namely with $V(H) = V(G)$ and $E(H) \subset E(G)$. Then $w(H) \geqslant w(G)$.*

Proof. If W is a watching system for H, then the same W is a watching system for G, since two adjacent vertices in H are also adjacent in G. □

Theorem 2.4. *[2] For all $n \geqslant 1$, we have*

$$w(P_n) = \lceil (n+1)/2 \rceil. \tag{3}$$

We have $w(C_4) = 3$, and for $n = 3$ and all $n \geqslant 5$

$$w(C_n) = \lceil n/2 \rceil. \tag{4}$$

3 Main Result

In this section we compute the watching number of Cayley graphs $Cay(Z_2 \times Z_\beta, S)$, where $\beta \in \{p, 2^\alpha, 3^\alpha, 5^\alpha, 7^\alpha\}$.

3.1 Watching number for Cayley graphs $Cay(Z_2 \times Z_p, S)$

We first show that the graph is a bipartite. Since the elements of the group are ordered pair $(0,i)$ and $(1,i)$, where $i \in \{0,1,...,p-1\}$, the graph has $2p$ vertices. On the other hand, since $\varphi_2 = 1$ and $\varphi_p = \{1,2,...,p-1\}$, The members of the S are in ordered pair by $\{(1,j) | j \in \{1,2,...,p-1\}\}$.

Now, if we consider each element as $(0,i)$, with elements such as $(1,k+i)$, where $k = 1,...,p-1$ and $i = 0,...p-1$ adjacent, according to the definition of the Cayley graphs and oprations group. On the contrary, consider each element as $(1,j)$ with elements such as $(0,k+j)$, where $k = 1,...,p-1$ and $j = 1,...p-1$ adjacent. Therefore, vertices of the set of vertices $\{(0,i) | i \in \{0,...,p-1\}\}$ and $\{(1,i) | i \in \{0,...,p-1\}\}$ are not adjacent and the same case $\{(0,i) | i \in \{0,...,p-1\}\} \cup \{(1,i) | i \in \{0,...,p-1\}\} = V(G)$. Therefore $Cay(Z_2 \times Z_p, S)$ is bipartite graph which each part has p vertices and the degree of each vertex is equal to $p-1$. In fact, $Cay(Z_2 \times Z_p, S)$ is complete bipartite graph, which a perfect matching is removed.

We consider two partitions of vertices by $A = \{(0,i) | i \in \{0,...,p-1\}\}$ and $B = \{(1,i) | i \in \{0,...,p-1\}\}$ and $p \neq 2^k - 1$. To cover the vertices of part B, we need at least $\lceil log(p) \rceil$ watchers in partition A. We place each watcher in the position vertex in part A. By placing these watchers on the vertices, all the vertices of the part B are covered. Now, it is enough to put $\lceil log(p - \lceil log p \rceil) \rceil$ watchers on the vertices of partition B, in which the vertices of partition A, which are not watchers, are covered by these watchers, and the such vertices are seen by the watchers that are in part B. Therefore all the vertices of the graph are covered with this watchers. So, we have

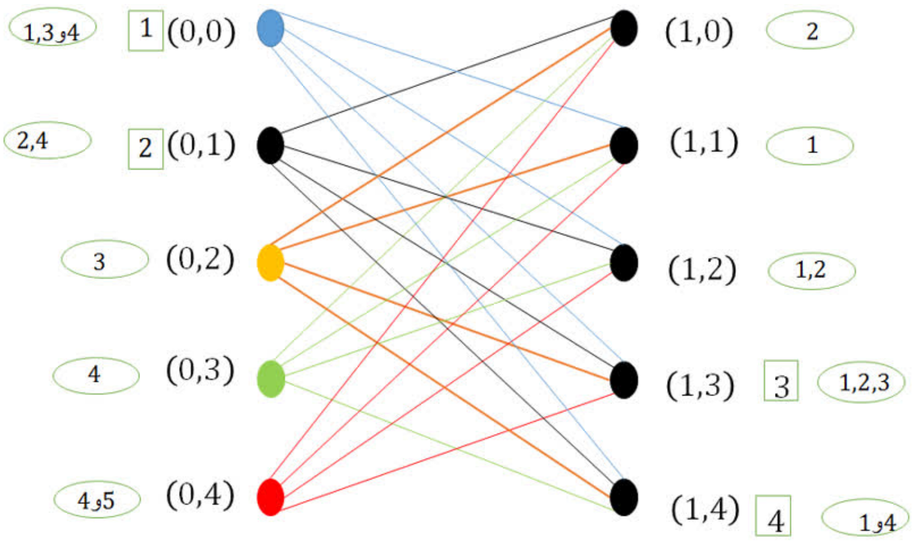

Figure 1. The Cayley Graph $(Z_2 \times Z_5, S)$.

$$w(Cay(Z_2 \times Z_p, S)) \leq \lceil log(p) \rceil + \lceil log(p - \lceil log(p) \rceil) \rceil. \quad (5)$$

For examole, suppose that $p = 5$, we want to get the watching numbers for cayley graph $Cay(Z_2 \times Z_5, S = \{(1,1),(1,2),(1,3),(1,4)\})$. We know that for cover the vertices of part B, we need at least $\lceil log(5) \rceil = 3$ watchers that we can put two of them in part 1 and one of them in part B, as shown in Figure 1.. Now we need to put $\lceil log(5 - \lceil log(5)) \rceil = 1$ watchers on the vertices of part B, to cover the vertices of part A, that are not watchers of them, and these vertices are seen by the watchers that are in part B. So, we have

$$w(Cay(Z_2 \times Z_5, S)) \leq \lceil log(5) \rceil + \lceil log(5 - \lceil log(5) \rceil) \rceil = 4. \quad (6)$$

On the other hand, when $p = 2^k - 1$, by above discussion, the vertices of part B are covered by $\lceil log(p) \rceil$ watchers on part A, and we need $\lceil log(p - \lceil logp \rceil) \rceil + 1$ watchers on part B to cover $p - \lceil logp \rceil$ vertices of the part A. Therefore all the vertices of the graph are covered with these watchers. So, we have

$$w(Cay(Z_2 \times Z_p, S)) \leq \lceil log(p) \rceil + \lceil log(p - \lceil log(p) \rceil) \rceil + 1. \quad (7)$$

For examole, suppose that $p = 7$, we want to get the watching numbers for Cayley graph $Cay(Z_2 \times Z_7, S = \{(1,1),(1,2),...,(1,6)\})$. We know that for cover the vertices of part B, we need at least $\lceil log(7) \rceil = 3$ watchers that we put them on the vertices of part A, as shown in Figure 2.. Now we need to put $\lceil log(7 - \lceil log(7)) \rceil + 1 = 3$ watchers on the vertices of part B, to cover the vertices of part A.

3.2 Watching number of Cayley graph $Cay(Z_2 \times Z_{2^\alpha}, S)$

The Cayley graph $Cay(Z_2 \times Z_{2^\alpha}, S)$, where $S = \varphi(2) \times \varphi(2^\alpha)$, is a disconnected graph with two connected components of orders $k_{2^{\alpha-1}}$ and $2^{\alpha-1}$. For this, suppose G_1 and G_2 are

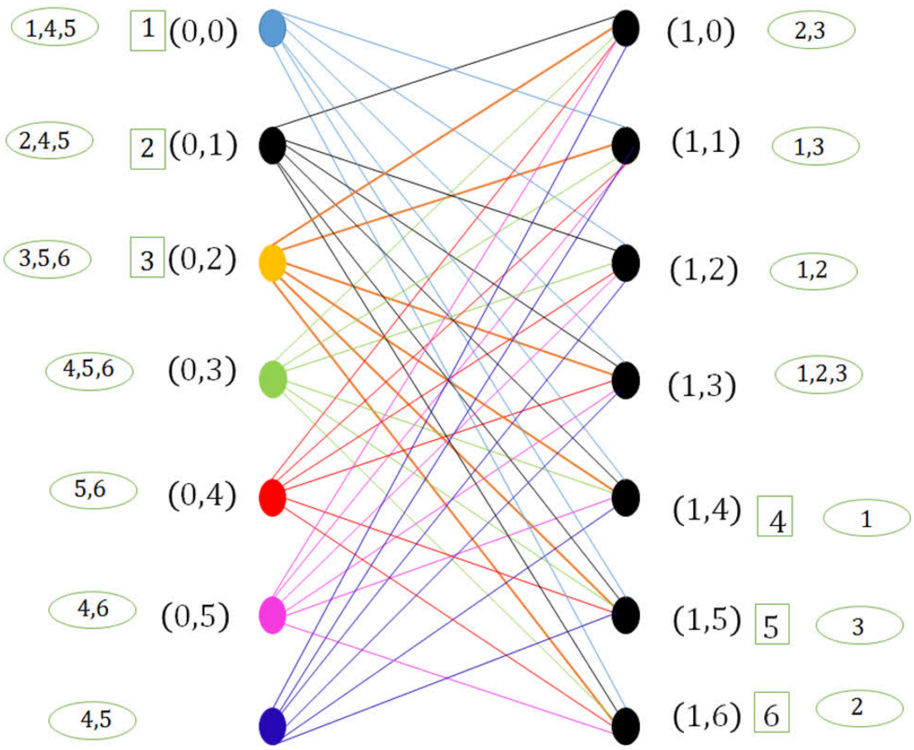

Figure 2. The Cayley Graph $(Z_2 \times Z_7, S)$.

the components, where $V(G_1) = \{(1,v)|v \text{ is odd }\} \cup \{(0,v)|v \text{ is even}\}$ and $V(G_2) = \{(0,v)|v \text{ is odd }\} \cup \{(1,v)|v \text{ is even}\}$. On other hand, every Cayley graph $Cay(G,S)$ is $|S|$-regular or equivalently $\varphi(p) = p-1$-regular, then $Cay(Z_2 \times Z_{2^\alpha}, S)$ is a disconnected graph with two connected components of orders $k_{p^{\alpha-1}}$ and $p^{\alpha-1}$. Since each connected component is covered by $\lceil log(2^{\alpha-1})\rceil + \lceil log(2^{\alpha-1} - \lceil 2^{\alpha-1}\rceil)\rceil$ watchers, and this graph has two connected components, we have

$$w(Cay(Z_2 \times Z_{2^\alpha}, S)) \leq 2 \times (\lceil log(2^{\alpha-1})\rceil + \lceil log(2^{\alpha-1} - \lceil log(2^{\alpha-1})\rceil)\rceil).$$

3.3 Watching number of Cayley graph $Cay(Z_2 \times Z_{3^\alpha}, S)$, where $\alpha \geq 2$

Similar to the last case this graph is bipartite. Let's show the set of vertices of the graph as set $A = \{(0,i)|i \in \{0,1,...,p-1\}\}$ and set $B = \{(1,i)|i \in \{0,1,...,p-1\}\}$. We know that $|S| = 2 \times 3^{\alpha-1}$, therefore the degree of each vertex in this graph is equal to $2 \times 3^{\alpha-1}$. Consider the vertices of one of the sets (for example, the set A). Observe that the vertices of this part can be divided into three subsets with elements $(o,3i)$, $(0,3i+1)$, and $(0,3i+2)$, so that the vertices of each subset have common neighbors and there are $3^{\alpha-1}$ common neighbors in each of the two distinct subsets without any common neighbors in three subsets. Therefore, by some vertices of two subsets of vertices of part A,(for example, $(0,3i)$ and $(0,3i+1)$) we must cover the vertices of a subset of vertices of part B(for example, $(1,3i+2)$). So, it's enough to put $\lceil log(3^{\alpha-1})\rceil$ watchers on some vertices of two subsets, so that the difference between watchers in each subset is at most one, this means that we put $\lceil \frac{(\lceil log 3^{\alpha-1})\rceil}{2}\rceil$ watchers

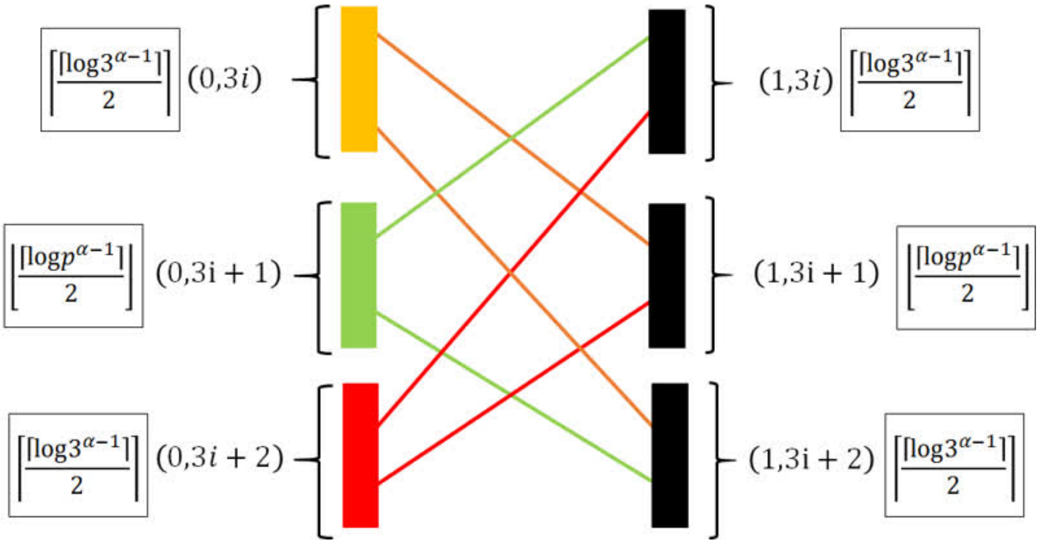

Figure 3. The Cayley Graph $(Z_2 \times Z_{3^{\alpha-1}}, S)$.

on one subset, and $\lfloor \frac{(\lceil log 3^{\alpha-1})\rceil}{2} \rfloor$ watchers on other sebset. Now, since the three subset of vertices in part A do not share any vertex in common, for cover the remaining vertices in part B, it is enough to put $\lceil \frac{(\lceil log 3^{\alpha-1})\rceil}{2} \rceil$ watchers on the third subset (Vertices with labels $(0,3i+2)$), see Figure 3.. Since the graph has two parts, we have

$$w(Cay(Z_2 \times Z_{3^\alpha}, S)) \leq 2 \times (\lceil log(3^{\alpha-1}) \rceil + \lceil \frac{\lceil log(3^{\alpha-1}) \rceil}{2} \rceil).$$

It is clear that if $\alpha = 1$, the graph is the same order with C_6 and we know that $w(C_6) = 3$.

3.4 Watching number of Cayley graph $Cay(Z_2 \times Z_{5^\alpha}, S)$, where $\alpha > 2$

Again, this graph is a bipartite and regular of degree $5^{\alpha-1}$. Observe that the vertices of every part can be divided into five subsets with elements $(k,5i)$, $(k,5i+1)$, $(k,5i+2)$, $(k,5i+3)$ and $(k,5i+4)$, where $k = 0$ or 1, and where there is no vertex in common. Also, there are $5^{\alpha-1}$ common neighbors in each four distinct subsets and $2 \times 5^{\alpha-1}$ common neighbors in each three distinct subsets and $3 \times 5^{\alpha-1}$ common neighbors in each two distinct subsets. Similar to last case we obtain

$$w(Cay(Z_2 \times Z_{5^\alpha}, S)) \leq 2 \times (\lceil log(5^{\alpha-1}) \rceil + \lceil \frac{\lceil log(5^{\alpha-1}) \rceil}{4} \rceil),$$

see Figure 4..

3.5 watching number for Cayley graphs $Cay(Z_2 \times Z_{7^\alpha}, S)$

First, let's assume that $\alpha \geq 6$, the graph is bipartite regular graph of degree $7^{\alpha-1}$. Since each part of the graph can be divided into seven subsets, in which the vertices of each subset have $6 \times 7^{\alpha-1}$ common neighbors. Since the vertices of each subset of the second part are seen by $6 \times 7^{\alpha-1}$ vertices, it is enough to put $(\lceil log(7^{\alpha-1}) \rceil$ watchers on the vertices of the six

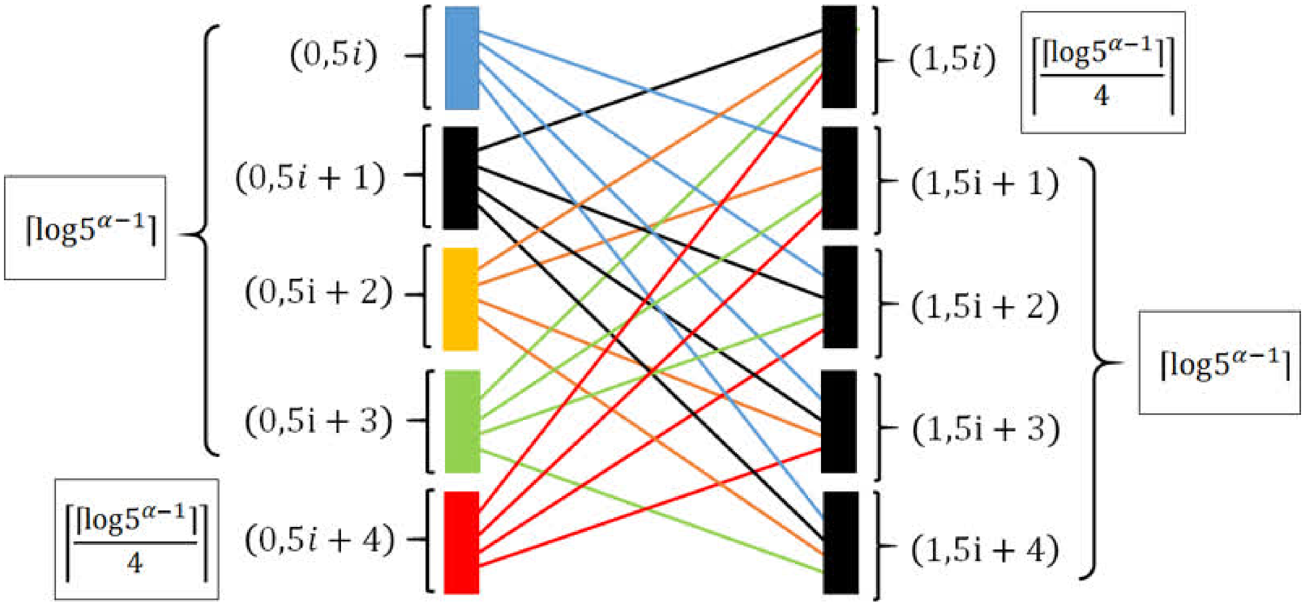

Figure 4. The Cayley Graph $(Z_2 \times Z_{5^{\alpha-1}}, S)$.

distinct subsets, where the difference watchers is at most one. Since the seven subsets in part A do not share any common neighbor, for cover the remaining vertices in part B, it is enough to put $\lceil \frac{\lceil log(7^{\alpha-1}) \rceil}{6} \rceil$ watchers on the vertices of sixth subset. Again, the graph has two parts and we yield

$$w(Cay(Z_2 \times Z_{7^\alpha}, S)) \leq 2 \times (\lceil log(7^{\alpha-1}) \rceil + \lceil \frac{\lceil log(7^{\alpha-1}) \rceil}{6} \rceil).$$

Now, suppose that $\alpha \leq 5$. By placing $\lceil log(7^\alpha) \rceil$ watchers on each part, where the difference number is at most one, the vertices of the second part are covered. With the same discussion, we put the same number of watchers on the second part, to cover the vertices of the first part. It is clear:

$$w(Cay(Z_2 \times Z_{7^\alpha}, S)) \leq 2 \times (\lceil log(7^\alpha) \rceil).$$

3.6 Watching number of Cayley Graphs $Cay(D_{2n}, S)$

Suppose that D_{2n} is a dihedral group with the following presentation

$$D_{2n} = \{s^n = k^2 = 1, k^{-1}sk = s^{-1}\}.$$

Here, we try to obtain a suitable bound for the Cayley graph $Cay(D_{2n}, S)$, where$|S| = 3$, and following properties:

First, assume that $S = \{x^i, x^{-i}, y\}$. Following two cases hold:

Case1. If x^i is a generator of group, then the graph is connected 3−regular and of the same order with $C_n \times K_2$. In this case, we can cnclude

$$w(C_n \times K_2) \leq n, \quad (8)$$

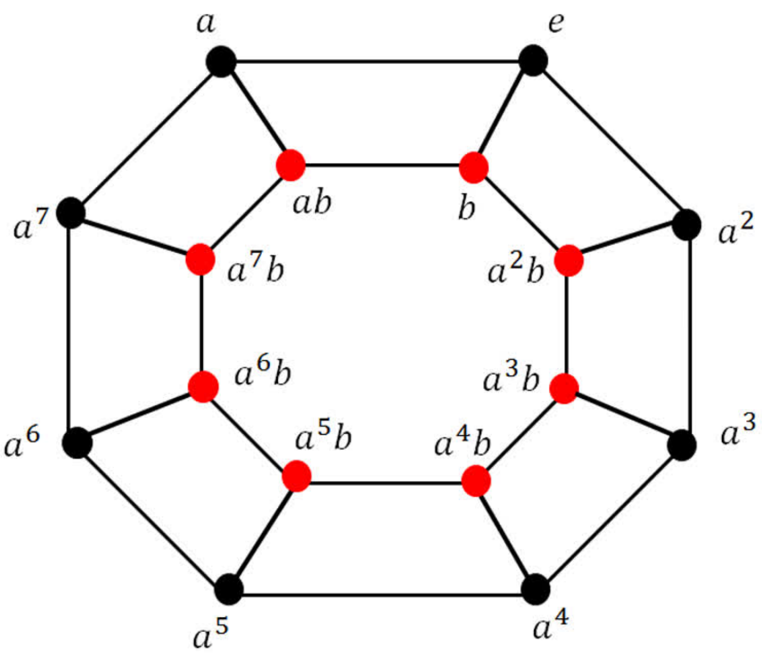

Figure 5. The Cayley Graph $(D_{16},(\{x^i,x^{-i},y\}))$.

or equivalently

$$w(Cay(D_{2n},(\{x^i,x^{-i},y\}))) \leq n,$$

see Fiqure 5..

Case 2. Suppose x^i is not a generator of D_{2n}. Thus, the graph is 3−regular, disconnected graph of the same order components. It is well-known that

$$\frac{o(x^i)}{(i,n)} = j. \tag{9}$$

Hence, the graph has i components of the same order with $C_j \times K_2$. Sinse $w(C_j \times K_2) \leq j$, we obtain

$$w(Cay(D_2,(\{x^i,x^{-i},y\}))) \leq i \times j.$$

Suppose now $S = \{x^iy, x^{-i}y, y\})$. If n is a prime number, the graph is a connected 3−regular and bipartit graph. Put watchers on vertices of one part of the grah. We have

$$w(Cay(D_{2n},(\{x^i,x^{-i},y\}))) \leq n.$$

If n is not a prime number, the graph is 3−regular, disconnected graph with two components of the same order. If we assume

$$\frac{o(x^iy)}{(i,n)} = j, \tag{10}$$

then G has i components with the same order, and each component is a bipartite graph. Hence

$$w(Cay(D_2,(\{x^i y, x^{-i} y, y\}))) \leq n,$$

as we desired.

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