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Research Paper

A unified approach to the incidence graphs of (weak) generalized quadrangles

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Abstract. A generalized quadrangle is a point-line geometry such that the incidence graph is a connected, bipartite graph of diameter 4 and girth 8. In this paper, we investigate the connection between generalized quadrangles and octographic bipartite graph (shortly, $\mathcal{O}$-graph), which are a class of bipartite graphs satisfying certain axioms regarding graph-theoretic properties of them. We prove that every incidence graph of a generalized quadrangle is a $\mathcal{O}$-graph. Also we obtain some properties of $\mathcal{O}$-graphs in terms of graph invariants. Finally, we conclude by discussing the implications of our findings and potential avenues for future research in this area.

Keywords. Generalized quadrangles, finite geometries, bipartite graphs

Mathematics Subject Classification (2010): 05B25; 51E12.

1 Introduction

Let $G = (V, E)$ be a simple graph where adjacency between vertices v_i and v_j is denoted by $v_i v_j \in E(G)$ or $v_i \sim v_j$. The distance between two vertices $u, v \in V(G)$ is denoted as $d(u, v)$. This metric encapsulates the shortest length of paths connecting the vertices u and v within the graph. The maximum eccentricity among the vertices of G is called the *diameter*, denoted by $diam(G)$, while the minimum eccentricity of its vertices is called the *radius*, denoted by

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$rad(G)$. Let N_{v_i} be the neighbor set of a vertex v_i in $V(G)$. The girth of a graph is defined as the length of the shortest cycle in the graph. In other words, it is the length of the smallest cycle in the graph. The common neighbour of vertices $v_1, v_2, \ldots v_k$ is denoted as $CN(v_1, \ldots, v_k)$, and its cardinality is denoted as $|CN(v_1, \ldots, v_k)| = cn(v_1, \ldots, v_k)$. A bipartite graph is a graph whose vertex set forms two disjoint sets in which no two vertices within the same set are adjacent. The Cartesian product of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, denoted by $G \times H$, is a graph with vertex set $V_G \times V_H$, where each vertex (u, v) is associated with a copy of G and a copy of H. The edge set of $G \times H$ consists of all pairs of edges $((u_1, v_1), (u_2, v_2))$ such that either $u_1 = u_2$ and there is an edge between v_1 and v_2 in H, or $v_1 = v_2$ and there is an edge between u_1 and u_2 in G. A grid graph is the Cartesian product of two paths, and it is denoted by $P_n \times P_m$, where P_n is a path on n vertices and P_m is a path on m vertices. The resulting grid graph has nm vertices and $2nm - n - m$ edges.

An incidence system of triples $\Gamma = (P, L, I)$ is called a point-line geometry, where P and L denote two sets of elements referred to as points and lines, respectively, and $I \subseteq (P \times L) \cup (L \times P)$ is a symmetric relation. A generalized n-gon is a point-line geometry such that the incidence graph is a connected, bipartite graph of diameter n and girth $2n$. [1]. Also there are different types of definition of generalized n-gon. (see also [3–5]). Tits [2] presents that a generalized quadrangle is a generalized 4-gon. According to Payne et.al. [13], a generalized quadrangle manifests as a symmetrical structure that conforms to a point-line incidence relation I, characterized by the following axioms:

- Every point is incident with exactly $1 + t$ lines, where $t \geq 1$, and any two distinct points are incident with at most one line.
- Every line is incident with exactly $1 + s$ points, where $s \geq 1$, and any two distinct lines are incident with at most one point.
- For any given point x and line L not incident with x, there exists a unique pair (y, z) of points and lines respectively, such that x is incident with z, z is incident with y, and y is incident with L.

The analogue definition of GQ is given by Maldeghem [5] as the following:
A weak generalized quadrangle (P, L, I) is a point-line incidence geometry satisfying the following axioms:

- Every point is incident with at least two, but not all lines.
- Every line is incident with at least two, but not all points
- For every point x and every line L not incident with x, there is a unique incidence point-line pair (y, M) with $LIyIMIx$.

Moreover, a (thick) generalized quadrangle is a weak generalized quadrangle satisfying the additional following axiom:

- Every point is incident with at least three lines and every line is incident with at least three points.

In the study of finite geometries, generalized quadrangles play a significant role due to their connections to many areas of mathematics, including group theory, algebraic geometry, and combinatorics. It is also well known that the incidence graph of a generalized quadrangle is a bipartite graph that encodes the incidence relations between points and lines of the geometry. Hence, generalized quadrangles have been extensively studied in graph theory, particularly in the context of distance-regular graphs, strongly regular graphs, and other related combinatorial objects (see also [6–12]).

Given a generalized quadrangle, it can be more complicated to show the incidence graph of it. Instead of that concept, the simpler concept can be useful in understanding the properties of the incidence graph of a generalized quadrangle. Thus we define a special type of bipartite graph, called a bipartite octographic graph (shortly $\mathcal{O}$-graph), has emerged as a useful tool in the study of finite geometries. We investigate the connection between incidence graphs of generalized quadrangles and $\mathcal{O}$-graphs. We show that an $\mathcal{O}$-graph is also a graph of diameter 4 and girth 8 and obtain that incidence graph of a generalized quadrangle is always a $\mathcal{O}$-graph. Specifically, we give a construction of $\mathcal{O}$-graph and prove that every incidence graph of grid graphs is also $\mathcal{O}$-graph.

2 Main Results

Definition 1. Let $G = (V, E)$ be a bipartite graph with $V = U \cup W$. The octographic bipartite graph is a special bipartite graph satisfying the following axioms hold:

($\mathcal{C}_1$.) For every $w \in W$, the degree of vertex w is greater than or equal 2.

($\mathcal{C}_2$.) For every $u_1, u_2 \in U$, the number of common neighbours of u_1 and u_2, denoted by $cn(u1, u2)$, is less than or equal to 1.

($\mathcal{C}_3$.) For $u \in U$, $w \in W$ such that $uw \notin E$, there is exactly one vertex x in $N(u)$ such that $cn(x, w) = 1$, where $N(u)$ denotes the set of neighbours of u.

($\mathcal{C}_4$.) There exists at least one pair of vertices x, y in U (or in W) such that $cn(x, y) = 0$.

Througout this paper we will use the notation $\mathcal{O}$-graph for the octographic bipartite graph.

Remark 1. In Fig1, there are given two GQ's. The first graph is 2×3 Grid graph. But it can not be $\mathcal{O}$- graph. Indeed, the vertices 1 and 5 have two common neighbours, thus it does not satisfy the axiom $\mathcal{C}_2$ in Definition 1. Similarly, Doily graph shows a generalized quadrangle, but it is not $\mathcal{O}$-graph.

Lemma 2.1. *Let $G = (U \cup W, E)$ be a octographic bipartite graph. For every pair of vertices p, q in G, we have $cn(p, q) \leq 1$.*

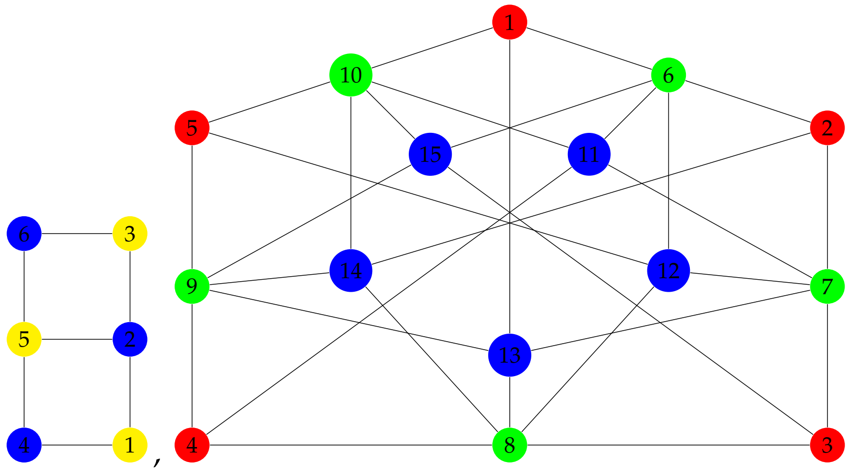

Figure 1. A 2×3 Grid and Doily Graph

Proof. If $p,q \in U$, then by $(\mathcal{C}_2)$, $cn(p,q) \leq 1$. Now assume that $p,q \in W$ and $cn(p,q) \geq 2$. Then there exist at least two vertices in U, say u_i and u_j, such that $p,q \in CN(u_i,u_j)$. Thus, $cn(u_i,u_j) \geq 2$, which contradicts $(\mathcal{C}_2)$. Therefore, $cn(p,q) \leq 1$. Let $p \in U$ and $q \in W$ and $cn(p,q) \geq 1$. Then there exists at least one vertex $x \in CN(p,q)$, which is a neighbour of both p and q. Hence, $x \in U \cap W$, which contradicts the assumption that $U \cap W = \emptyset$. Therefore, $cn(p,q) = 0$.

Thus, for every pair of vertices p and q in G, $cn(p,q) \in \{0,1\}$. □

Lemma 2.2. *Let $G = (U \cup W, E)$ be a octographic bipartite graph with $\delta(G) \geq 2$. In this case,*

i. For any $u_1,u_2 \in U$, we have $d(u_1,u_2) \in \{2,4\}$.

ii. For any $w_1,w_2 \in W$, we have $d(w_1,w_2) \in \{2,4\}$.

iii. For any $u \in U$ and $w \in W$, we have $d(u,w) \in \{1,3\}$.

Proof.

i. Let G be a $\mathcal{O}$-graph. In this case, due to $(\mathcal{C}_2)$, for all $u_1,u_2 \in U$, $cn(u_1,u_2) \leq 1$.

Case 1: Let $cn(u_1,u_2) = 0$. Then $\delta(G) \geq 2$, so there exist distinct $w_1,w_2 \in N(u_1)$ and $w_3,w_4 \in N(u_2)$ such that $w_1 \neq w_2$ and $w_3 \neq w_4$. By $(\mathcal{C}_2)$, $u_1w_3 \notin E$, so there must be at least one $u_3 \in U$ such that $CN(w_1,w_3) = \{u_3\}$. A shortest path between u_1 and u_2 is $u_1 - w_1 - u_3 - w_3 - u_2$, which has a length of 4.

Case 2: Let $cn(u_1,u_2) = 1$. Then there exists a $w \in W$ such that $CN(u_1,u_2) = \{w\}$. In this case, a shortest path between u_1 and u_2 is simply $u_1 - w - u_2$, so $d(u_1,u_2) = 2$.

ii. Since G is a $\mathcal{O}$-graph, by Lemma 2.1, for all $uv \in G$, $cn(u,v) \leq 1$.

Let $cn(w_1,w_2) = 0$ for $w_1,w_2 \in W$. Then there exist distinct nodes $u_1,u_2,u_3,u_4 \in U$ such that $N(w_1) = \{u_1,u_2\}$ and $N(w_2) = \{u_3,u_4\}$ since $\delta(G) \geq 2$. Due to $(\mathcal{C}_3)$, $u_1w_2 \notin E$, so there must be a vertex $u_5 \in U$ such that $CN(w_2,w_3) = \{u_5\}$, where w_3 is a neighbour of u_1. The shortest path between w_1 and w_2 is $w_1 - u_1 - w_3 - u_5 - w_2$, which has a length of 4.

Let $cn(w_1,w_2) = 1$. Then there exists a node $u \in U$ such that $N(w_1,w_2) = \{u\}$. The shortest path between w_1 and w_2 is $w_1 - u - w_2$, which has a length of 2.

iii. Since G is a $\mathcal{O}$-graph, for $u \in U$ and $w \in W$, there are two cases. In case of $u \in N(w)$, then $d(u,w) = 1$. Now Let $u \notin N(w)$ and $uw \notin E$, then due to $(\mathcal{C}_3)$, there exists a unique vertex $v \in W$ with $CN(w,v) = \{k\}$ and $u \neq k$. So, the shortest path between u and w is $u - v - k - w$, which has a length of 3. □

Corollary 2.3. *Let G be a $\mathcal{O}$-graph. Then $diam(G) = 4$.*

Proof. By Lemma 2.2, it is easy to see that no pair of vertices has distance greater than 4. Hence $diam(G)$ is 4.

□

Theorem 2.4. *The girth of a $\mathcal{O}$-graph is* 8.

Proof. Now, assume that the girth of a Q graph is less than 8. This means that the shortest cycle in the graph has length at most 6. Since the graph is bipartite, the shortest cycle must have an even length. Therefore, the shortest cycle in the graph has length 4 or 6.

Let $girth(G) = 4$. Then G contains an $u_1 - w_1 - u_2 - w_2 - u_1$ cycle such that $u_1, u_2 \in U$ and $w_1, w_2 \in W$. In this case, since $w_1, w_2 \in CN(u_1, u_2)$, we get $cn(u_1, u_2) \geq 2$. Hence this contradicts axiom $(\mathcal{C}_2)$.

Let $girth(G) = 6$ and let $u_1 - w_1 - u_2 - w_2 - u_3 - w_3 - u_1$ be a cycle of length 6 such that $u_1, u_2, u_3 \in U$ and $w_1, w_2, w_3 \in W$. Hence for $u_1 w_2 \notin E$, there is a w_1 in $N(u_1)$ such that $cn(w_1, w_2) = 0$. But this contradicts the axiom $(\mathcal{C}_3)$. Therefore, the assumption that the girth of the $\mathcal{O}$-graph is less than 8 leads to a contradiction. Hence, the girth of a $\mathcal{O}$-graph graph is 8. □

Proposition 2.5. *There is no $\mathcal{O}$-graph of order $n \leq 7$.*

Proof. First, note that for a $\mathcal{O}$-graph G with n vertices, the size of the vertex set can be written as $n = 2k + 1$, where k is a positive integer. This is because the existence of a pair of vertices in U or W with no common (as required by the fourth axiom of $\mathcal{O}$-graphs) implies that $|U| \geq 2$ and $|W| \geq 2$.

Now, suppose there exists an $\mathcal{O}$-graph G of order ≤ 7. Since n must be odd, we have $n = 3, 5$, or 7. We consider each case separately:

Case 1: $n = 3$.

Since G has only three vertices, it must be a tree or a cycle. However, neither a tree nor a cycle satisfies the second axiom of $\mathcal{O}$-graphs.

Case 2: $n = 5$

In this case, G must have two vertices in one partition and three in the other (assume two vertices in U and three in W). Since G is a bipartite graph, the only possibility is a tree with two vertices of degree 1 in W. However, this graph violates the first axiom of $\mathcal{O}$-graphs.

Case 3: $n = 7$

Suppose G has two vertices in U and five vertices in W. By the pigeon-hole principle, there must be at least three vertices in W with a common neighbour in U. But this contradicts the second axiom of $\mathcal{O}$-graphs. Therefore, there is no $\mathcal{O}$-graph of order ≤ 7. □

Theorem 2.6. *The $\mathcal{O}$-graph of order 8 is isomorphic to cycle C_8. That's, there is no $\mathcal{O}$-graph of order 8 except C_8.*

Proof. First, note that a $\mathcal{O}$-graph of order 8 must have 4 vertices in each partition U and W. Also, by axiom $(\mathcal{C}_1)$, every vertex in W must have degree at least 2. Suppose there exists a $\mathcal{O}$-graph G of order 8 that is not isomorphic to C_8. We will show that G cannot satisfy at least one of axioms of $\mathcal{O}$-graphs.

Since G is not isomorphic to C_8, it must have a vertex of degree greater than 3. By axiom $(\mathcal{C}_1)$, this vertex must be in W. Without loss of generality, let this vertex be $w_1 \in W$.

By axiom $(\mathcal{C}_3)$, for any $u \in U$ such that $uw_1 \notin E$, there exists exactly one vertex $x \in N(u)$ such that $cn(x, w_1) = 1$. Without loss of generality, let $u_1, u_2 \in U$ be two non-adjacent vertices such that u_1 is adjacent to w_1 and u_2 is not. Then, u_1 and u_2 must have a common neighbour x such that $cn(x, w_1) = 1$. But this contradicts axiom $(\mathcal{C}_2)$, which requires that the number of common neighbours of any two vertices in U be at most 1.

Therefore, our assumption that there exists a $\mathcal{O}$-graph of order 8 that is not isomorphic to C_8 leads to a contradiction, and we can conclude that there is no $\mathcal{O}$-graph of order 8 except C_8. □

Remark 2. Let G be a $\mathcal{O}$-graph. Then there are at least four vertices in U such that $cn(u_i, u_j, u_k, u_l) = 0$. Moreover, there is at least four vertices in W such that $cn(w_i, w_j, w_k, w_l) = 0$. Therefore, we get that $|U| \geq 4$ and $|W| \geq 4$.

Theorem 2.7. *The incidence graph of a (weak) generalized quadrangle is an $\mathcal{O}$-graph.*

Proof. The incidence graph of a (weak) generalized quadrangle is a bipartite graph whose vertices are the points and lines of the quadrangle, with two vertices adjacent if and only if the corresponding point and line intersect.

Let w be a vertex in W, representing a line in the generalized quadrangle. Since any two lines intersect in at most one point in a (weak) generalized quadrangle . Therefore, the degree of the vertex w must be greater than or equal to 2, since it must be incident with at least two points. So, axiom $(\mathcal{C}_1)$ holds.

Let $u_1, u_2 \in U$, representing two points in the (weak) generalized quadrangle. Since any two points lie on at most one line in a (weak) generalized quadrangle. Therefore, the number of common neighbours of u_1 and u_2 in the incidence graph is either 0 or 1. So, axiom $(\mathcal{C}_2)$ holds.

Let $u \in U$ and $w \in W$ be such that $uw \notin E$. Since there is exactly one line passing through the point u and meeting line w in a (weak) generalized quadrangle. Let x be the point at which this line intersects with another line incident with u, which is guaranteed to exist by the third axiom ($\mathcal{C}_3$). Then, x is the unique vertex in $N(u)$ such that $cn(x,w) = 1$.

In a (weak) generalized quadrangle, there exist four points, no three of which are collinear. Let x and y be two non-collinear points. Then, there is no line incident with both x and y, so $cn(x,y) = 0$. Hence axiom ($\mathcal{C}_4$) is satisfied.

Therefore, the incidence graph of a (weak) generalized quadrangle is an $\mathcal{O}$-graph. □

Remark 3. The above theorem gives that incidence graphs of generalized quadrangles are also $\mathcal{O}$-graph. Since all grids are (weak) GQ, $\mathcal{O}$ graph can be consructed by the incidence graph of the grid graphs. For instance, the incidence graph of 2×3 Grid graph gives the $\mathcal{O}$ graph of order 11 (See Figure 2).

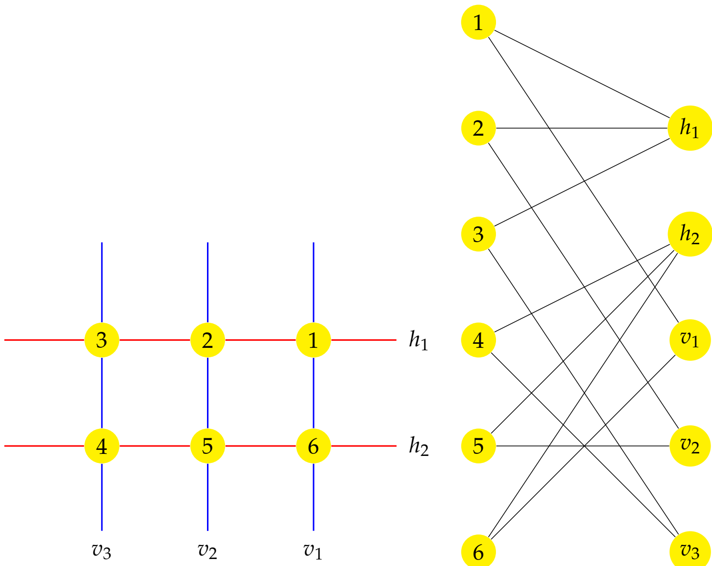

Figure 2. 2×3 Grid and its incidence graph

Conclusion

In this paper, we investigate the connection between incidence graphs of generalized quadrangles and $\mathcal{O}$-graphs. We begin by providing a comprehensive introduction to generalized quadrangles and their incidence graphs, as well as $\mathcal{O}$-graphs and their properties. Then, we explore the conditions under which an incidence graph of a generalized quadrangle is an $\mathcal{O}$-graph. Our results shed light on the relationship between these two types of bipartite graphs and provide a new perspective on the structure of generalized quadrangles.

The definition of $\mathcal{O}$-graph provides a way to study certain types of bipartite graphs with specific properties. By defining the axioms that an $\mathcal{O}$-graph must satisfy, we can identify and study these graphs in a more systematic way. This can help in understanding the structural properties of $\mathcal{O}$-graphs, their relationships to other types of graphs, and their applications in various fields. Additionally, the definition of $\mathcal{O}$-graphs allows us to establish certain results and theorems that apply specifically to this type of graph.

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