Journal of Discrete Mathematics and Its Applications 9 (1) (2024) 51–64

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Research Paper

Perfect State Transfer on Cayley Graphs over Groups

Afsaneh Khalilipour, Modjtaba Ghorbani*

Department of Mathematics, Faculty of Science, Shahid Rajaee Teacher Training University, Lavizan, Tehran, I. R. Iran

Academic Editor: Majid Arezoomand

Abstract. Perfect state transfer (PST) on graphs due to their significant applications in quantum information processing and quantum computations. In the present work, we establish a characterization of Cayley graphs over U_{6n} group, having PST.

Keywords. Linear transformation, perfect state transfer, matrix representation, Cayley graph.

Mathematics Subject Classification (2010): 05C25, 81P45, 81Q35.

1 Introduction

Perfect state transfer (PST) is a quantum phenomenon in which a quantum state can be transferred from one place to another without losing information. Let Γ be an undirected simple graph whose vertex set is denoted by V(Γ) and A= A(Γ) be the adjacency matrix of Γ. For a real number t, the *transfer matrix* of Γ is defined as the following $n \times n$ matrix:

$$H(t) = H_\Gamma(t) = \exp(-itA) = \sum_{s=0}^{+\infty} \frac{(-itA)^s}{s!} = (H(t))_{u,v \in V(\Gamma)},$$

where $i = \sqrt{-1}$ and $n = |V(\Gamma)|$ is the number of vertices in Γ. Therefore, we have the decomposition of the transfer matrix

$$H(t) = \exp(-i\lambda_1 t)E_1 + \cdots + \exp(-i\lambda_n t)E_n.$$

*Corresponding author (*Email address*: mghorbani@sru.ac.ir)
Received 20 January 2024; Revised 08 February 2024; Accepted 22 February 2024
First Publish Date: 1 March 2024

Definition 1. Let Γ be a graph. For two distinct vertices $u, v \in V(\Gamma)$, we say that Γ has a perfect state transfer (PST) from u to v at the time $t(>0)$ if the (u,v)-entry of $H(t)$, denoted by $H(t)_{u,v}$, has absolute value 1. We say that Γ is periodic at u with period t if $H(t)_{u,u}$ has absolute value 1. If Γ is periodic with period t at every point, then Γ is said to be periodic.

The occurrence of PST in quantum communication networks was first introduced by Bose in [10]. This idea has since gained significant research interest due to its many applications in quantum information processing, as highlighted in [1–3, 9, 13] and other related works. He proposed the idea of using spin chains to transfer quantum states over short distances and showed that the highest fidelity is obtained for short spin chains (number of spins ~ 100). In [14], Christandl and et al. indecated that perfect quantum state transfer between antipodal points of a N-link hypercubes occurs if and only if $N \leq 3$. Bašić in [8] using the circulant graph proved that PST exists between two distinct vertices a and b in the spin network, whenever $\tau \in \mathbb{R}^+$ exist that $|F(\tau)_{a,b}| = 1$ where $F(t) = \exp(iAt)$ and A is the circulant graph adjacency matrix. Saxena and et al. in [28] proved that if there is the time $\tau \in \mathbb{R}^+$, that for each vertex a of the graph, $|F(\tau)_{a,a}| = 1$ if and only if all eigenvalues of the graph are integers. In the integral circulant graph $ICG_n(D)$ with set of vertices $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, Two vertices a and b are adjacent whenever $\gcd(a-b, n) \in D$ which $D = \{d \,|\, d \mid n, 1 \leq d \leq n\}$. Godsil in [19, 20], investigated the necessary conditions for the occurrence of PST and its applications in cryptographic systems. In [19], he studied the results of PST according to algebraic graph theory with the help of properties of the function $\exp(itA)$ where A is the adjacency matrix of the graph. In particular, he showed in [20] that if PST occurs in a graph, then the square of its spectral radius is either an integer or lies in a quadratic extension of the rationals. As a result, for any integer k, there are only finitely many graphs with maximum valency k on which PST occurs. He also showed that if PST happens from vertex u to vertex v, then the graphs $\Gamma \setminus u$ and $\Gamma \setminus v$ are cospectral and any automorphism Γ that fixes u, must fix v (and conversely). Coutinho and Godsil in [15] studied graphs whose adjacency matrix is the sum of tensor products of 01-matrices, focusing on the case where a graph is the tensor product of two other graphs. As a result, they constructed many new ones that have PST. Tan and et al. in [32] presented a characterization of the occurrence of PST in connected simple Cayley graphs $\Gamma = \mathrm{Cay(G,S)}$, where G is an abelian group and S is a non-empty subset of G. They showed that many previous results about periodicity and existence of PST in circulant graphs (where the underlying group G is cyclic) and cubelike graphs $G = (\mathbb{F}_{2n}, +)$ can be obtained or extended to arbitrary abelian cases in a unified and simpler way by using their description. Cao and et al. in [11] investigated the existence of PST in the Cayley graph $\mathrm{Cay(D_n,S)}$ with non-normal S. They demonstrated that if n is odd, then the Cayley graph $\mathrm{Cay(D_n,S)}$ does not possess PST and for even integers n, it is proved that if $\mathrm{Cay(D_n,S)}$ has PST, then S is normal. Cao and Feng in [12], investigated the existence of PST on Cayley graphs over dihedral groups. They proved that if n is odd integer and S is a conjugation-closed subset of the dihedral group D_n, then the Cayley graph $\mathrm{Cay(D_n,S)}$ does not possess PST. They also presented specific constructions for even integers n where $\mathrm{Cay(D_n,S)}$ has PST. Luo and et al. in [26] studied the existence of PST on

Cayley graphs over semi-dihedral groups which are non-abelian groups. Using the representations of semi-dihedral groups, they provided some necessary and sufficient conditions for Cayley graphs over semi-dihedral groups admitting *PST*. Applying those conditions, they provided examples of Cayley graphs over semi-dihedral groups that exhibit *PST*. Additionally, they proposed results regarding whether certain new Cayley graphs over non-abelian groups possess *PST*. Arezoomand and et al. in [7] established a characterization of Cayley graphs over dicyclic groups T_{4n} that possess *PST*. As a consequence of their main result, they investigated the existence of *PST* on a quasiabelin Cayley graph $\mathrm{Cay}(\mathrm{T_{4n}},\mathrm{S})$. In the same year, in [5], he gave a characterization of Cayley graphs over groups with an abelian subgroup of index 2 having *PST*, which improves upon earlier results regarding Cayley graphs over abelian groups, dihedral groups, dicyclic group and determined Cayley graphs over generalized dihedral groups and generalized dicyclic groups having *PST*. Wang and et al. in [33], established the necessary and sufficient condition for a bi-Cayley graph having perfect state transfer over any given finite abelian group. As a result of this work, numerous known and new findings regarding Cayley graphs with *PST* over abelian groups, generalized dihedral groups, semi-dihedral groups, and generalized Quaternion groups were obtained as corollaries. Notably, they presented an example of a connected non-normal Cayley graph over a dihedral group that exhibits *PST* between two distinct vertices, which was previously thought impossible. In this paper, we explore the existence of *PST* on Cayley graph over U_{6n} group.

The rest of the paper is organized as follows. In Section 2, we present some comments on notations used in this paper. In Lemma 2.4, we first give a description of the representations of U_{6n}. Next, in Proposition 2.2 and Corollary 2.3, we investigate a general method for computing the spectra and its corresponding eigenvectors of Cayley graphs over finite groups. We then apply this method to compute the eigenvectors of U_{6n}. Finally, the existence of *PST* on the Cayley graph $\mathrm{Cay}(\mathrm{U_{6n}},\mathrm{S})$ is studied in Section 3.

2 Preliminaries

In this section, we will review some standard facts and notation used throughout this paper. Our notation for representations of finite groups is based on [22].

2.1 The representations of U_{6n} and spectra of $\mathrm{Cay}(U_{6n},S)$

Let G be a finite group, $\mathbb{C}$ the field of complex numbers, and V a $\mathbb{C}$-vector space with dimension $n < \infty$ over $\mathbb{C}$. A $\mathbb{C}$-representation of G is a group homomorphism $\xi : G \to GL(V)$ for non-zero vector space V. The dimension of V is called the degree of ξ. If β a $\mathbb{C}$-basis for V, then $[\xi]_\beta$ a matrix $\mathbb{C}$-representation of G into the multiplicative group of non-singular $n \times n$ matrices over $\mathbb{C}$.
We say that if $\varrho : G \to GL(n,\mathbb{C})$ is a representation of the group G, then the vector space V by definition of the multiplication $vg = v(g\varrho)$ for every $v \in V$ and $g \in G$, such that vg lies

in V, it becomes a G-module over the complex numbers $\mathbb{C}$. An $\mathbb{C}G$-module V is said to be irreducible if it is non-zero and it has no $\mathbb{C}G$-submodule except $\{0\}$ and V. A representation ϱ is irreducible if the corresponding $\mathbb{C}G$-module V is irreducible. If V isreducible then ϱ is reducible, it means a representation of the form

$$g \mapsto \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix},$$

where $A(g) \in M_{n_1}(\mathbb{C}), B(g) \in M_{n_1 \times n_2}(\mathbb{C}), C(g) \in M_{n_2}(\mathbb{C})$, and where $n_1, n_2 \in \mathbb{N}$ are independent of $g \in G$. If V_1 and V_2 are isomorphic, then there is an isomorphism $T : V_1 \to V_2$ satisfying $[\xi]_{\beta_1}(g) = T^{-1}[\zeta]_{\beta_2}(g)T$, where $[\xi]_{\beta_1}$ a $\mathbb{C}$-representation of G on V_1 and $[\zeta]_{\beta_2}$ a $\mathbb{C}$-representation of G on V_2. So ξ and ζ be equivalent representations of the group G over $\mathbb{C}$. Every $\mathbb{C}$-representation ϱ of G gives rise to a character $\chi : G \to \mathbb{C}$, which is defined as the function $\chi(g) = \mathrm{tr}(\varrho(g))$, where $\mathrm{tr}(\varrho(g))$ denotes the trace of the matrix $\mathbb{C}$-representation ϱ of G on V. We say that χ is an irreducible character of G if χ is the character of an irreducible representation.
The conjugate-transpose or adjoint of A is the matrix $A^* = \overline{A^T}$. A matrix $A \in \mathrm{GL}_n(\mathbb{C})$ is unitary if and only if $A^{-1} = A^*$. The unitary $n \times n$ matrices form a subgroup $U_n(\mathbb{C})$ of $\mathrm{GL}_n(\mathbb{C})$. A representation $\varrho : G \to \mathrm{GL}_n(\mathbb{C})$ is said to be unitary if $\varrho(g) \in \mathrm{U}(n)$ for all $g \in G$. By [29, Proposition 3.2.4], every complex representation $\varrho : G \to \mathrm{GL}_n(\mathbb{C})$ is equivalent to a unitary representation.

A circulant matrix is a type of square matrix that follows a very distinct pattern: each row is the same as the previous row, only rotated one unit to the right. That is, each row is a circular variation of the first row. In this paper which we showe by $\mathrm{C}(c_0, c_1, \cdots, c_n)$. Additionally, we note that if each row of the square matrix is obtained by one left-shifting of the previous row, an anti-circulant matrix.
The eigenvalues of a graph Γ are defined to be the eigenvalues of its adjacency matrix $A(\Gamma)$. The scalar $\lambda \in \mathbb{C}$ is said to be an eigenvalue of A, if there exists a non-zero vector $x \in \mathbb{C}^n$ such that $Ax=\lambda x$. Therefore x is the eigenvector corresponding to the eigenvalue of λ. The set of all eigenvalues of Γ is called the spectrum of Γ. Since $A(\Gamma)$ is a real symmetric matrix, the eigenvalues of Γ, denoted as λ_i $(i = 1,2,...,n)$, are real numbers.
let a and b be elements of G. We say that a is conjugate to b in G, if there exists an element $g \in G$ such that $g^{-1}ag = b$. The set of all elements in G that are conjugate to a is denoted as a^G and is defined as $\{g^{-1}ag : g \in G\}$. This set is called the conjugacy class of a in G. If a and b are two conjugate elements of group G, then for all characters χ of G, we have $\chi(a) = \chi(b)$.

Assume that $n \geq 1$ is an integer. Define the U_{6n} group by $\langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$. It should be noted that group $U_{6n} = \{a^k, a^k b, a^k b^2 \mid 0 \leq k \leq 2n-1\}$ has $3n$ conjugacy classes as the following:

$$\{a^{2j}\}, \{a^{2j}b, a^{2j}b^2\}, \{a^{2j+1}, a^{2j+1}b, a^{2j+1}b^2\} \quad (0 \leq j \leq n-1).$$

Lemma 2.1. *[24] Let $n \geq 1$, $\varepsilon = e^{2\pi i/2n} = \cos(\pi/n) + i\sin(\pi/n)$ be a $2n$-th root of unity which is neither 1 nor -1. The irreducible representations and characters of U_{6n} is listed in the Tables 1. and 2.*

Table 1. Irreducible representation of U_{6n}, $\omega = e^{2\pi i/3}$

	a^{2j}	a^{2j+1}	$a^{2j}b$
$\varphi_k(0 \leq k \leq 2n-1)$	ε^{2kj}	$\varepsilon^{k(2j+1)}$	ε^{2kj}
$\gamma_l(0 \leq l \leq n-1)$	$\begin{pmatrix} \varepsilon^{2lj} & 0 \\ 0 & \varepsilon^{2lj} \end{pmatrix}$	$\begin{pmatrix} 0 & \varepsilon^{l(2j+1)} \\ \varepsilon^{l(2j+1)} & 0 \end{pmatrix}$	$\begin{pmatrix} \varepsilon^{2lj}\omega & 0 \\ 0 & \varepsilon^{2lj}\omega^2 \end{pmatrix}$

Table 2. Character Table of U_{6n}.

	a^{2j}	a^{2j+1}	$a^{2j}b$
$\chi_k(0 \leq k \leq 2n-1)$	ε^{2kj}	$\varepsilon^{k(2j+1)}$	ε^{2kj}
$\psi_l(0 \leq l \leq n-1)$	$2\varepsilon^{2lj}$	0	$-\varepsilon^{2lj}$

In general, using the subsequent result, one can compute spectra and eigenspaces of Cayley graphs over finite groups. Given an irreducible representation or a character ψ of a group G and a subset S of G, we denote $\sum_{s\in S}\psi(s)$ by $\psi(S)$. Let S be a symmetric subset of a finite group G, this means that $S = S^{-1}$. We assume that the identity element of G is not belonged to S, i.e. $1_G \notin S$. The Cayley graph of G with respect to S, $\mathrm{Cay}(G,S)$, is the graph whose vertices are the elements of G and there exists an edge between different vertices $g,h \in G$ if $gh^{-1} \in S$. If S is a normal Cayley set in G, i.e. $g^{-1}Sg = S$ for each $g \in G$. then we call Γ a *quasiabelian Cayley graph* of G with respect to S (see [31]). Note that, since S is symmetric and $1_G \notin S$, $\Gamma = \mathrm{Cay}(G,S)$ is a simple graph. The adjacency matrix of Γ is defined by $A = A(\Gamma) = (a_{g,h})_{g,h\in G}$ where

$$a_{g,h} = \begin{cases} 1 & \text{if } gh^{-1} \in E(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Remember that $V(\Gamma)$ and $E(\Gamma)$ stand for the set of vertices and edges of Γ, respectively. One can refer to [23] for more properties about Cayley graphs.

Proposition 2.2. *[6, Corollary 7 and Lemma 11] Let $\Gamma = \mathrm{Cay}(G,S)$ be an undirected Cayley graph over a finite group G with irreducible unitary matrix representations $\varrho^{(1)},\ldots,\varrho^{(m)}$. Let d_l be the degree of $\varrho^{(l)}$. For each $l \in \{1,\ldots,m\}$, define a $d_l \times d_l$ block matrix $A_l := \varrho^{(l)}(S)$. Let $\chi_{A_l}(\lambda)$ and $\chi_A(\lambda)$ be the characteristic polynomial of A_l and A, the adjacency matrix of Γ, respectively. Then*

(1) there exists a basis $\mathcal{B}$ such that $[A]_{\mathcal{B}} = \mathrm{Diag}(A_1 \otimes I_{d_1},\ldots,A_m \otimes I_{d_m})$.

(2) $\chi_A(\lambda) = \Pi_{l=1}^{m} \chi_{A_l}(\lambda)^{d_l}$.

(3) *Let $v_{(k)}$ be an eigenvector of A_k, $1 \leq k \leq m$, associated with λ. Then the following vectors are distinct linearly independent d_k eigenvectors of Γ associated with λ $v_{(k)}^{j} := \sum_{g \in G} \left[v_{(k)} \cdot \varrho_j^{(k)}(g) \right] e_g$, $(1 \leq j \leq d_k)$ where $\cdot$ is the usual inner product and $\varrho_j^{(k)}(g)$ is a vector whose coordinates are the coordinates of jth column of $\varrho^{(k)}(g)$.*

Corollary 2.3. *[7, Corollary 3.3] Keeping the notations of Proposition 2.2 and considering fixed ordering $g_1 = 1, g_2, \cdots, g_n$ of all elements of G, we have*

1. *Let $U = (v_{(k)}^{j})^T$ and $U.U^* = [u_{rs}]$. Then*
$$u_{rs} = [v_{(k)}.\varrho_j^{(k)}(g_r)][\bar{v}_{(k)}.\bar{\varrho}_j^{(k)}(g_s)].$$
2. *If $\varrho^{(k)}$ is 1-dimensional representation of G, then $\lambda = \varrho^{(k)}(S)$ is an eigenvalue of A_k,$v_{(k)} = 1$ and $v_{(k)}^{1} = \sum_{g \in G} \varrho^{(k)}(g) e_g$ is an eigenvector of Γ associated to the eigenvalue $\varrho^{(k)}(S)$. Furthermore, by the above notation $u_{rs} = \varrho^{(k)}(g_r)\bar{\varrho}^{(k)}(g_s) = \varrho^{(k)}(g_r {g_s}^{-1})$.*
3. *If for every $g \in G$, we have that $\sum_{s \in S} \varrho^{(k)}(gsg^{-1}) = \sum_{s \in S} \varrho^{(k)}(s)$, then $\lambda_k = \frac{\chi_k(S)}{d_k} = \frac{\sum_{s \in S} \chi(s)}{d_k}$ is an eigenvalues of Γ with multiplicity ${d_k}^2$ and standard basis $e_1, e_2, \cdots e_{d_k}$ are eigenvectors of A_k associated to $\lambda_k = \frac{\chi_k(S)}{d_k}$. Furthermore, the eigenvectors $v_{ij}^{(k)} = \sqrt{\frac{d_k}{|G|}} \sum_{g \in G} \varrho_{ij}^{(k)}(g) e_g = \sqrt{\frac{d_k}{|G|}} (\varrho_{ij}^{(k)}(g_1), \cdots, \varrho_{ij}^{(k)}(g_n))$, $1 \leq i, j \leq d_k$, which are associated to λ_k form an orthonormal basis for the eigenspace V_{λ_k}, where $\varrho_{ij}^{(k)}(g)$ is the ij-entry of the matrix $\varrho^{(k)}(g)$. Also, by the notation of (1), we have u_{rs}=$\varrho_{ij}^{(k)}(g_r)\bar{\varrho_{ij}}^{(k)}(g_s) = \varrho_{ij}^{(k)}(g_r g_s^{-1})$.*

The succeeding lemma which appears as [29, Exercise 5.12.3] is a direct consequence of the last part of Corollary 2.3 and determines eigenvalues and eigenvectors of adjacency matrix of a Cayley graph $\Gamma = \mathrm{Cay}(G, S)$, where S is conjugation-closed, namely $gSg^{-1} = S$ for all $g \in G$.

Lemma 2.4. *[29, Exercise 5.12.3] Let $G = \{g_1, \cdots, g_n\}$ be a finite group of order n and $\varphi^{(1)}, \cdots, \varphi^{(t)}$ be a complete set of unitary representatives of the equivalence classes of irreducible representations of G. Let χ_i be the character of $\varphi^{(i)}$ and d_i be the degree of $\varphi^{(i)}$. Let $S \subseteq G$ be a symmetric set and assume further that S is conjugation-closed. Then the eigenvalues of the adjacency matrix A of the Cayley graph of $Cay(G, S)$ with respect to S are $\lambda_1, \cdots, \lambda_t$, where*
$$\lambda_k = \frac{1}{d_k} \sum_{s \in S} \chi_k(s), \;\; 1 \leq k \leq t,$$
and that λ_k has multiplicity d_k^2. Moreover, the vectors
$$v_{ij}^{(k)} = \sqrt{\frac{d_k}{|G|}} (\varphi_{ij}^{(k)}(g_1), \cdots, \varphi_{ij}^{(k)}(g_n))^T, \;\; 1 \leq i, j \leq d_k$$
form an orthonormal basis for the eigenspace V_{λ_k} associated with λ_k.

Consider the group U_{6n} ($n \geq 1$), and the Cayley graph $\Gamma = \mathrm{Cay}(U_{6n}, S)$, where S is a symmetric subset of U_{6n}. We regard the conjugacy classes of U_{6n} as follows:

$A_0 = \{1_{U_{6n}}\}$, $A_r = \{a^{2r}\}$ $(1 \leq r \leq n-1)$, $B = \{a^{2s}b, a^{2s}b^2 : 0 \leq s \leq n-1\}$,

$C = \{a^{2s+1}, a^{2s+1}b, a^{2s+1}b^2 : 0 \leq s \leq n-1\}$.

Let $A = (\bigcup_{r=1}^{n-1} A_r)$, $S = S_A \cup S_B \cup S_C$, where $S_X = S \cap X$ for $X \in \{A, B, C\}$.

We suppose that $|S| = s$, $|S_A| = s_a$, $|S_B| = s_b$, $|S_C| = s_c$. $s = s_a + s_b + s_c$.

First, we consider the one-dimensional representations of U_{6n}. Applying Corollary 2.3 (part 1), the adjacency matrix of the Cayley graph $\mathrm{Cay}(U_{6n}, S)$ has the following eigenvalues and eigenvectors:
for $0 \leq k \leq 2n-1$,

$\lambda_{\chi_k} = \sum_{a^{2j} \in S} \varepsilon^{2kj} + \sum_{a^{2j}b, a^{2j}b^2 \in S} \varepsilon^{2kj} + \sum_{a^{2j+1}, a^{2j+1}b, a^{2j+1}b^2 \in S} \varepsilon^{k(2j+1)}$,

$v_{\chi_k} = \frac{1}{\sqrt{6n}} (\{\varepsilon^{2kj}\}_{j=0}^{n-1}, \{\varepsilon^{2kj}, \varepsilon^{2kj}\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)k}, \varepsilon^{(2j+1)k}, \varepsilon^{(2j+1)k}\right\}_{j=0}^{n-1})^T$.

Now suppose that γ_r is a two-dimensional irreducible representation of U_{6n} for $0 \leq l \leq n-1$. Then $\gamma_l(S) = \gamma_l(S_A) + \gamma_l(S_B) + \gamma_l(S_C)$, where

$$\gamma_l(S_A) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \gamma_l(S_B) = \begin{pmatrix} -(y_1+y_2) & 0 \\ 0 & -(y_1+y_2) \end{pmatrix}, \gamma_r(S_C) = \begin{pmatrix} 0 & z-(z_1+z_2) \\ z-(z_1+z_2) & 0 \end{pmatrix}.$$

and
$x = \sum_{a^{2j} \in S} \varepsilon^{2lj}$, $y_1 = \sum_{a^{2j}b \in S} \varepsilon^{2lj}$, $y_2 = \sum_{a^{2j}b^2 \in S} \varepsilon^{2lj}$, $z = \sum_{a^{2j+1} \in S} \varepsilon^{(2j+1)l}$, $z_1 = \sum_{a^{2j+1}b \in S} \varepsilon^{(2j+1)l}$,

$z_2 = \sum_{a^{(2j+1)}b^2 \in S} \varepsilon^{(2j+1)l}$.
Hence $\gamma_l(S) = \begin{pmatrix} x-y & z-z' \\ z-z' & x-y \end{pmatrix}$, where $y = y_1 + y_2$ and $z' = z_1 + z_2$.

Therefore the eigenvalues of $\Gamma = \mathrm{Cay}(G, S)$ corresponding to the two dimensional unitary representations γ_l of U_{6n} for $l = 0, 1, \cdots, n-1$ are $\lambda_{l_1} = (x-y) + (z-z')$ and $\lambda_{l_2} = (x-y) - (z-z')$. On the other hand, the eigenvectors of $\gamma_l(S)$ corresponding to eigenvalue $\lambda \in \{\lambda_{l_1}, \lambda_{l_2}\}$ are $v_{\lambda_{l_1}} = (1, \ell_{l_1})$ and $v_{\lambda_{l_2}} = (\ell_{l_2}, 1)$. Now by applying Proposition 2.2, the associated eigenvectors of $\mathrm{Cay}(G, S)$ corresponding to the eigenvalue $\lambda \in \{\lambda_{l_1}, \lambda_{l_2}\}$ are $v_\lambda^{(j)} = \sum_{g \in G} [v_\lambda \cdot {\gamma_l}^j(g)] e_g$ $(j = 1, 2)$, where ${\gamma_l}^j(g)$ is the jth column-vector of $\gamma_l(g)$. Therefore, we have the following eigenvectors:

$$v_{l_1}^{(1)} = (\left\{\varepsilon^{2lj}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{2lj}\omega\right\}_{j=0}^{n-1}, \left\{\varepsilon^{2lj}\omega^2\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\ell_{l_1}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega\ell_{l_1}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega^2\ell_{l_1}\right\}_{j=0}^{n-1})^T,$$

$$v_{l_1}^{(2)} = (\left\{\varepsilon^{2lj}\ell_{l_1}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{2lj}\omega^2\ell_{l_1}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{2lj}\omega\ell_{l_1}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega^2\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega^2\right\}_{j=0}^{n-1})^T,$$

$$v_{l_2}^{(1)} = (\left\{\varepsilon^{2lj}\ell_{l_2}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{2lj}\omega\ell_{l_2}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{2lj}\omega^2\ell_{l_2}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega^2\right\}_{j=0}^{n-1})^T,$$

$$v_{l_2}^{(2)} = (\left\{\varepsilon^{2lj}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{2lj}\omega^2\right\}_{j=0}^{n-1}, \left\{\varepsilon^{2lj}\omega\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\ell_{l_2}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega^2\ell_{l_2}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega^2\ell_{l_2}\right\}_{j=0}^{n-1})^T.$$

Notice that, for any $0 \le j \le n-1$, the sets $\{v_{l_1}^{(1)}, v_{l_1}^{(2)}\}$ and $\{v_{l_2}^{(1)}, v_{l_2}^{(2)}\}$ are the orthogonal bases for the eigenspaces $V_{\lambda_{l_1}}$ and $V_{\lambda_{l_2}}$ associated with λ_{l_1} and λ_{l_2}, respectively. Furthermore, one can assume that $\langle v_{l_1}^{(1)}, v_{l_2}^{(1)} \rangle = 0$ since we have an inner product on V, and thus $\bar{\ell}_{l_2} = -\ell_{l_1}$. Now let $\ell_l = \ell_{l_1}$ and $\iota_l = 1 + |\ell_l|^2$. Thus we have the following eigenvectors:

$$v_{l_1}^{(1)} = \tfrac{1}{\sqrt{3n\iota_l}}(\{\varepsilon^{2lj}\}_{j=0}^{n-1}, \{\varepsilon^{2lj}\omega\}_{j=0}^{n-1}, \{\varepsilon^{2lj}\omega^2\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\ell_l\right\}_{j=0}^{n-1}, \varepsilon^{(2j+1)l}\omega\ell_l\Big\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega^2\ell_l\right\}_{j=0}^{n-1})^T,$$

$$v_{l_1}^{(2)} = \tfrac{1}{\sqrt{3n\iota_l}}(\{\varepsilon^{2lj}\ell_l\}_{j=0}^{n-1}, \{\varepsilon^{2lj}\omega^2\ell_l\}_{j=0}^{n-1}, \{\varepsilon^{2lj}\omega\ell_l\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega^2\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega^2\right\}_{j=0}^{n-1})^T,$$

$$v_{l_2}^{(1)} = \tfrac{1}{\sqrt{3n\iota_l}}(\{-\varepsilon^{2lj}\bar{\ell}_l\}_{j=0}^{n-1}, \{-\varepsilon^{2lj}\omega\bar{\ell}_l\}_{j=0}^{n-1}, \{-\varepsilon^{2lj}\omega^2\bar{\ell}_l\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega\right\}_{j=0}^{n-1}, \left\{\varepsilon^{(2j+1)l}\omega^2\right\}_{j=0}^{n-1})^T,$$

$$v_{l_2}^{(2)} = \tfrac{1}{\sqrt{3n\iota_l}}(\{\varepsilon^{2lj}\}_{j=0}^{n-1}, \{\varepsilon^{2lj}\omega^2\}_{j=0}^{n-1}, \{\varepsilon^{2lj}\omega\}_{j=0}^{n-1}, \left\{-\varepsilon^{(2j+1)l}\bar{\ell}_l\right\}_{j=0}^{n-1}, \left\{-\varepsilon^{(2j+1)l}\omega^2\bar{\ell}_l\right\}_{j=0}^{n-1}, \left\{-\varepsilon^{(2j+1)l}\omega^2\bar{\ell}_l\right\}_{j=0}^{n-1})^T.$$

3 *PST* on Cayley Graphs

For a simple graph Γ with n vertices, $\mathrm{Spec}(\Gamma)$ denotes the set of all eigenvalues of Γ. For any symmetric matrix A, assume that its eigenvalues are λ_i's for $1 \leq i \leq n$. There is a unitary matrix $P = (v_1, \cdots, v_n)$, where each v_i is an eigenvector of $\lambda_i, (1 \leq i \leq n)$. Thus we have the following spectral decomposition of A

$$A = \lambda_1 E_1 + \cdots + \lambda_n E_n,$$

where $E_i = v_i {v_i}^* (1 \leq i \leq n)$ satisfies

$$E_i E_j = \begin{cases} E_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let Γ be an undirected simple graph whose vertex set is denoted by $V(\Gamma)$ and $A = A(\Gamma)$ be the adjacency matrix of Γ. For a real number t, the *transfer matrix* of Γ is defined as the following $n \times n$ matrix:

$$H(t) = H_\Gamma(t) = \exp(-itA) = \sum_{s=0}^{+\infty} \frac{(-itA)^s}{s!} = (H(t))_{u,v \in V(\Gamma)},$$

where $i = \sqrt{-1}$ and $n = |V(\Gamma)|$ is the number of vertices in Γ. Therefore, we have the decomposition of the transfer matrix

$$H(t) = \exp(-i\lambda_1 t) E_1 + \cdots + \exp(-i\lambda_n t) E_n.$$

] We also need notation of the 2-adic exponential valuation of rational numbers which is a mapping defined by

$$\eta_2 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}, \;\; \eta_2(0) = \infty, \;\; \eta_2(2^t \frac{a}{b}) = t, \quad \textit{where } a, b, t \in \mathbb{Z} \textit{ and } 2 \nmid ab.$$

We assume that $\infty + \infty = \infty + t = \infty$ and $\infty > t$ for any $t \in \mathbb{Z}$. Then for $\beta, \beta' \in \mathbb{Q}$, the following properties yield for η_2:

1. $\eta_2(\beta\beta') = \eta_2(\beta) + \eta_2(\beta')$,
2. $\eta_2(\beta + \beta') \geq \min(\eta_2(\beta), \eta_2(\beta'))$ and the equality holds if $\eta_2(\beta) \neq \eta_2(\beta')$.

3.1 *PST* on Cayley Graph over U_{6n} Group

Assume that $A = A(\Gamma)$ is the adjacency matrix of the Cayley graph $\mathrm{Cay}(U_{6n}, S)$. A has the eigenvectors v_k for $0 \leq k \leq 2n-1$ and $v_l^{(i)}$ for $1 \leq i \leq 4$ and $0 \leq l \leq n-1$, which are introduced in Section 2 depending on the parity of n. Hence we have the following unitary matrix:

$$P = (v_0, v_1, \cdots, v_{2n-1}, v_0^{(1)}, v_0^{(2)}, v_0^{(3)}, v_0^{(4)}, \cdots, v_l^{(1)}, v_l^{(2)}, v_l^{(3)}, v_l^{(4)}).$$

The corresponding projective matrices are $E_k = \frac{1}{6n}\begin{pmatrix} \Omega \otimes J_3 & \varepsilon^{-2j}\Omega \otimes J_3 \\ \varepsilon^{2j}\Omega \otimes J_3 & \Omega \otimes J_3 \end{pmatrix}$,

where J_m is the all-one matrix of order m, and Ω is the circulant matrix with the first row $(1,\varepsilon^{-2j},\varepsilon^{-4j},\cdots,\varepsilon^{-2j(n-1)})$. And

$$E_l^{(1)} = \frac{1}{3n}\begin{pmatrix} \Lambda & \bar{\varepsilon^l}\bar{\ell}_l\Lambda \\ \bar{\varepsilon^l}\bar{\ell}_l\Lambda & |\ell_l|^2\Lambda \end{pmatrix},\ E_l^{(2)} = \frac{1}{3n}\begin{pmatrix} |\ell_r|^2\Lambda^T & \varepsilon^{-l}\ell_l\Lambda^T \\ \bar{\varepsilon^{-l}}\bar{\ell}_l\Lambda^T & \Lambda^T \end{pmatrix},$$

$$E_l^{(3)} = \frac{1}{3n}\begin{pmatrix} |\ell_r|^2\Lambda & -\bar{\varepsilon^l}\bar{\ell}_l\Lambda \\ -\varepsilon^l\ell_l\Lambda & \Lambda \end{pmatrix},\ E_l^{(4)} = \frac{1}{3n}\begin{pmatrix} \Lambda^T & -\varepsilon^{-l}\ell_l\Lambda^T \\ -\bar{\varepsilon^{-l}}\bar{\ell}_l\Lambda^T & |\ell_l|^2\Lambda^T \end{pmatrix},$$

where Λ is the circulant matrix with the first row $(\Omega',\omega^2\Omega',\omega\Omega')$ and Ω' is the anti-circulant matrix the first row $(1,\varepsilon^{-2j},\varepsilon^{-4j},\cdots,\varepsilon^{-2j(n-1)})$.
Now we compute the (u,v)-th entry of the transfer matrix. We obtain:

1. *if* $0 \le u,v \le 3n-1$ *or* $3n \le u,v \le 6n-1$,

$$\begin{aligned}(H(t))_{u,v} = &\frac{1}{6n}(\sum_{k=0}^{2n-1}\frac{1}{\iota_l}(\varepsilon^{-(u-v)k})\exp(-i\lambda_k t)) \\ &+\frac{1}{3n}(\sum_{\substack{l=0\\0\le i,j\le 2}}^{n-1}\frac{1}{\iota_l}(\varepsilon^{-(u-v)l}\omega^{(i+j)}(1+|\ell_l|^2)\exp(-i\lambda_{l_1}t)) \\ &+\frac{1}{3n}(\sum_{\substack{l=0\\0\le i,j\le 2}}^{n-1}\frac{1}{\iota_l}(\varepsilon^{-(u-v)l}\omega^{(i+j)}(1+|\ell_l|^2)\exp(-i\lambda_{l_2}t)),\end{aligned} \tag{1}$$

2. *if* $0 \le u \le 3n-1$, $3n \le v \le 6n-1$,

$$\begin{aligned}(H(t))_{u,v} = &\frac{1}{6n}(\sum_{k=0}^{2n-1}\varepsilon^{-2k}.\varepsilon^{-(u-v)k})\exp(-i\lambda_k t) \\ &+\frac{1}{3n}(\sum_{\substack{l=0\\0\le i,j\le 2}}^{n-1}\frac{1}{\iota_l}(\bar{\varepsilon^l}\bar{\ell}_l.\varepsilon^{-(u-v)l}\omega^{i+j}+\varepsilon^l\ell_l.\varepsilon^{-(u-v)l}\omega^{i+j})\exp(-i\lambda_{l_1}t)) \\ &+\frac{1}{3n}(\sum_{\substack{l=0\\0\le i,j\le 2}}^{n-1}\frac{1}{\iota_l}(-\bar{\varepsilon^l}\bar{\ell}_l.\varepsilon^{-(u-v)l}\omega^{i+j}-\varepsilon^l\ell_l.\varepsilon^{-(u-v)l}\omega^{i+j})\exp(-i\lambda_{l_2}t)).\end{aligned} \tag{2}$$

3. *if* $3n \le u \le 6n-1$, $0 \le v \le 3n-1$, *then*

$$\begin{aligned}(H(t))_{u,v} &= \frac{1}{6n}\Big(\sum_{k=0}^{2n-1} \varepsilon^{2k}.\varepsilon^{-(u-v)k}\Big)\exp(-i\lambda_k t)\\ &+ \frac{1}{3n}\Big(\sum_{\substack{l=0\\0\le i,j\le 2}}^{n-1} \frac{1}{\iota_l}(\varepsilon^l \ell_l.\varepsilon^{-(u-v)l}\omega^{i+j} + \bar{\varepsilon}^l \bar{\ell}_l.\varepsilon^{-(u-v)l}\omega^{i+j})\exp(-i\lambda_{l_1} t)\Big)\\ &+ \frac{1}{3n}\Big(\sum_{\substack{l=0\\0\le i,j\le 2}}^{n-1} \frac{1}{\iota_l}(-\varepsilon^l \ell_l.\varepsilon^{-(u-v)l}\omega^{i+j} - \bar{\varepsilon}^l \bar{\ell}_l.\varepsilon^{-(u-v)l}\omega^{i+j})\exp(-i\lambda_{l_2} t)\Big).\end{aligned} \tag{3}$$

Applying the above arguments, we give a complete characterization of existence of *PST* for quasiabelian Cayley graphs over U_{6n} groups.

Theorem 3.1. *Let* $\Gamma = \mathrm{Cay}(U_{6n}, S)$ *be a quasiabelian Cayley graph with respect to* S*. Then* Γ *has* $2n$ *(not necessarily distinct) eigenvalues which correspond to the one-dimensional representations* φ_k $(0 \le k \le 2n-1)$*, respectively, with one is* $\lambda_0 = |S|$ *and the* $2n-1$ *other eigenvalues which are denoted by* λ_k*, and some multiple eigenvalues corresponding to the two dimensional representations* γ_l $(0 \le l \le n-1)$*, which are denoted by* μ_l*. Moreover* Γ *is periodic if and only if it is integral. The minimum period of the vertices is* $2\pi/M$*, where* $M = gcd(\lambda - \lambda_0 : \lambda \in \mathrm{Spec}(\Gamma) \setminus \{\lambda_0\})$*. Furthermore, For each* n*,* Γ *has PST between two vertices* u *and* v *if and only if*

1. *all eigenvalues of* Γ *are integers, namely,* Γ *is integral,*
2. $v = u + n$ *when* $0 \le u, v \le 3n-1$ *or* $3n \le u, v \le 6n-1$*,*
3. $\eta_2(\lambda_1 - \lambda_0)$*,* $\eta_2(\lambda_{2k'+1} - \lambda_0)$ *and* $\eta_2(\mu_{2l'-1} - \lambda_0)$ *are the same for all* $1 \le k' \le n-1$ *and* $1 \le l' \le n/2$*, say,* α*, and* $\eta_2(\mu_{2l'} - \lambda_0)$ *and* $\eta_2(\lambda_{2k'} - \lambda_0)$ *are bigger than* α *for all* $1 \le l' \le n/2$ *and* $1 \le k' \le n-1$*.*

Proof. Applying Corollary 2.3, we have the following unitary matrix

$$P = (v_0, v_1, \cdots, v_{2n-1}, v_0^{(1)}, v_0^{(2)}, v_0^{(3)}, v_0^{(4)}, \cdots, v_l^{(1)}, v_l^{(2)}, v_l^{(3)}, v_l^{(4)}).$$

where v_k and $v_l^{(i)}$ are introduced in Lemma 2.4 respectively $0 \le k \le 2n-1$, $1 \le i \le 4$ and $0 \le l \le n-1$. Since S is a normal Cayley subset, we have $\ell_l = \ell_{l_1} = \ell_{l_2} = 0$ and then $\iota_l = 1$. Now by applying the Section 3.1 when $\ell_l = 0$, we may obtain that: if $0 \le u, v \le 3n-1$ or $3n \le u, v \le 6n-1$, then

$$\begin{aligned}(H(t))_{u,v} &= \frac{1}{6n}\Big(\sum_{k=0}^{2n-1} (\varepsilon^{-(u-v)k})\exp(-i\lambda_k t)\Big)\\ &+ \frac{2}{3n}\Big(\sum_{\substack{l=0\\0\le i,j\le 2}}^{n-1} (\varepsilon^{-(u-v)l}\omega^{(i+j)})\exp(-i\mu_l t)\end{aligned}$$

Now it is immediate that;

$$\begin{aligned}|(H(t))_{u,v}| &\leq \frac{1}{6n}(\sum_{k=0}^{2n-1}|\varepsilon^{-(u-v)k}|)|\exp(-i\lambda_k t)| \\ &+\frac{2}{3n}(\sum_{\substack{l=0\\0\leq i,j\leq 2}}^{n-1}(|\varepsilon^{-(u-v)l}\omega^{(i+j)}|)|\exp(-i\mu_l t)| \\ &=\frac{2n}{6n}+\frac{2}{3n}\sum_{l=0}^{n-1}1=1\end{aligned}$$

Thus $|(H(t))_{u,v}|=1$ if and only if for $0\leq l\leq n-1$ and $1\leq k\leq 2n-1$, it holds that

$$\exp(-i\lambda_0 t)=\varepsilon^{-(u-v)k}\exp(-i\lambda_k t)=\varepsilon^{-(u-v)l}\omega^{(i+j)}\exp(-i\mu_l t),$$

where $i+j$ should be 3 or 0, otherwise $\omega^1+\omega^2=-1$. From the last two equations, since ε is the $2n$-th root of unity, we get that n divides $u-v$. Let $t=2\pi T$, we have that

$$\begin{aligned}&u-v\equiv 0 \pmod n,\\ &(\lambda_k-\lambda_0)T-\frac{k}{2}\in\mathbb{Z},\quad 1\leq k\leq 2n-1,\\ &(\mu_l-\lambda_0)T-\frac{l}{2}\in\mathbb{Z},\quad 0\leq l\leq n-1,\end{aligned}$$

Since $0=\mathrm{trA}=\sum_{k=0}^{2n-1}\lambda_k+4\sum_{l=0}^{n-1}\mu_l$, we have that $6n\lambda_0 T\in\mathbb{Z}$, and since $\lambda_0=|S|$ is a positive integer, we may conclude that $T\in\mathbb{Q}$. By using this fact that all the eigenvalues are algebraic number, and since in this case they are rational, we obtain that they are integral. Thus $\eta_2(\lambda_1-\lambda_0)$, $\eta_2(\lambda_{2k'+1}-\lambda_0)$ and $\eta_2(\mu_{2l'-1}-\lambda_0)$ for all $1\leq k'\leq n-1$ and $0\leq l'\leq n/2$, are a constant, say α, and $\eta_2(\mu_{2l'}-\lambda_0)$ and $\eta_2(\lambda_{2k'}-\lambda_0)$ are bigger than α for all $1\leq l'\leq n/2$ and $1\leq k'\leq n-1$.

□

Example 3.2. *Let $\Gamma=\mathrm{Cay}(U_{6n},S)$, where $S=\{a^{(2j+1)},a^{(2j+1)}b,a^{(2j+1)}b^2|\ 0\leq j\leq n-1\}$. Then, in both cases of even n and odd n, eigenvalues of Γ are $\begin{pmatrix}3n & 0 & -3n\\ 1 & 6n-2 & 1\end{pmatrix}$. Thus by Theorem 3.1, Γ is an integral graph and periodic with minimum period $2\pi/3n$, but it has no PST.*

Example 3.3. *Suppose n is an even integer and $S=U_{6n}\setminus\{1,a^n\}$ be a subset of U_{6n} such that $gSg^{-1}=S$ for all $g\in U_{6n}$. Let $\Gamma=\mathrm{Cay}(U_{6n},S)$, be the Cayley graph with connection set S. Then from the character table of U_{6n}, eigenvalues of Γ are $\lambda_0=6n-2$, $\lambda_1=0$ and for all $1\leq k'\leq n-1$, $1\leq l'\leq n/2$:*

$$\lambda_{2k'+1}=\mu_{2l'-1}=0\ \text{ and }\ \lambda_{2k'}=\mu_{2l'}=-2.$$

So Γ is integral. Hence it is periodic with minimum period π. Furthermore, $\eta_2(\lambda_1-\lambda_0)=1$, and for all $1\leq k'\leq n-1$ and $0\leq l'\leq n/2$ we have:

$$\eta_2(\lambda_{2k'+1}-\lambda_0)=\eta_2(\mu_{2l'-1}-\lambda_0)=1\ \text{ and }\ \eta_2(\lambda_{2k'}-\lambda_0)=\eta_2(\mu_{2l'}-\lambda_0)=2.$$

Then by Theorem 3.1 implies that Γ has PST between two vertices u and v, where $v=u+n$ when $0\leq u,v\leq 3n-1$ or $3n\leq u,v\leq 6n-1$.

4 Concluding remarks

We have presented a comprehensive survey of perfect state transfer (*PST*) on Cayley graphs over U_{6n} group. Our findings have significant implications for the development and optimization of quantum networks, as they provide graph structures that facilitate the efficient transfer of quantum information. Those also have broad implications for the design of quantum networks and suggest new avenues for quantum computing research. As more research is conducted in this area, it is likely that more insights and advances will emerge and advance our understanding of quantum systems and their applications in various disciplines. We are grateful to the reviewer for his careful reading and insightful comments that helped us improve this paper.

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Citation: A. Khalilipour, M. Ghorbani, Perfect state transfer on Cayley graphs over groups, J. Disc. Math. Appl. 9(1) (2024) 51–64.

https://doi.org/10.22061/jdma.2023.2073