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Research Paper

A generalized version of symmetric division degree index

Najaf Amraei*

Department of Mathematics, Faculty of Science, Shahid Rajaee Teacher Training University, Tehran, 16785-163, I. R. Iran

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Abstract. The symmetric division degree SDD-index of a simple connected graph G is defined as the sum of terms $f(d_u, d_v) = (d_u/d_v) + (d_v/d_u)$ over all pairs of distinct adjacent vertices of G; where d_u denotes the degree of a vertex u of graph G. In this paper, we introduce the general form of symmetric division degree index by replacing the degree of vertices $f(d_u, d_v)$ with another symmetric function of vertex properties. We establish some properties of the generalized symmetric division degree index $GSDD$ index for certain special functions and calculate the values of these new indices for some well-known graphs.

Keywords: Generalized symmetric division degree index, distance, total distance.

1 Introduction

In this paper, we consider a simple graph $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. An edge $e \in E(G)$ with end vertices u and v is denoted by $e = uv$. In a graph G, the neighborhood $N_G(v)$, or simply N_v, of a vertex v is the set of all vertices adjacent to v. The degree of a vertex v, d_v is the cardinality of N_v. We denote the smallest and largest degrees of graph G by $\delta(G)$ and $\Delta(G)$, respectively. If $\Delta(G) = \delta(G)$, then G is called a regular graph.

The distance $d(u, v)$ between two vertices u and v in a graph G is the length of the shortest path between them. The total distance $D(u)$ of a vertex u in a graph G is the sum of the

**Emial address*: n.amraei@yahoo.com
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distances between u and all other vertices in G, formally, $D(u) = \sum_{v \in V(G)} d(u,v)$.

The sum of the degrees of all the neighbors of the vertex u in the graph G is denoted by $Sd(u)$ in which $Sd(u) = \sum_{v \in N_u} d_v$.

Topological indices, including the Randić- type indices [11], Zagreb-type indices [8], Szeged index [8], and Wiener index and its modifications [15] are referred to as band-additive because they are presented as the sum of the contributions of the edges of the graph. These indices are expressed in the general form $\sum_{uv \in E(G)} f(u,v)$, where f may be a function of the vertex degree, the distance of a vertex from all other vertices in the graph, or other variables associated with the vertices.

In the graph theory, an edge-transitive graph is a graph G in which, given any two edges e and e' of G, there is some automorphism $f : G \to G$ such that$f(e) = e'$. The orbit of an edge $e \in E(G)$, denoted by E_e, is defined as $E_e = \{f(e) | f \in Aut(G)\}$. Similarly, a vertex-transitive graph can be defined. It is a well-known fact that all vertices in the same vertex-orbit have the same degree.

The symmetric division degree index, denoted by SDD, was introduced by Vukičević and Gašperov in [14]. It is defined as $SDD(G) = \sum_{uv \in E(G)} f(d_u, d_v)$, where $f(d_u, d_v) = f(d_v, d_u) = \frac{d_u}{d_v} + \frac{d_v}{d_u}$ and d_u denotes the degree of vertex u. For more details about this index see [1–7, 9, 10, 13] and the references cited therein. By replacing the degree of vertices with a function of vertex properties, we introduce a general form of the SDD is called as the generalized symmetric division degree, denoted by $GSSD$. The $GSSD$ of graph G with respect to function f is defined as follows:

$$GSSD_f(G) = \sum_{uv \in E(G)} \frac{f(u)}{f(v)} + \frac{f(v)}{f(u)}, \tag{1}$$

where $f(u)$ and $f(v)$ denote the values of function f at vertices u and v, respectively. See [12] for degree-based functions.

2 Some properties of generalized symmetric division degree index

The $GSSD$ index, provides a flexible and powerful tool for studying the structural properties of graphs. Its application to special functions and well-known graphs can yield valuable insights into the behavior of the index and its relationship to other topological indices.

The new index exhibits certain general properties that can be verified for any chosen f-function. In the following, we will obtaine some of these properties.

Theorem 2.1. *Let G be a graph with m edges and let f be a real-valued function on vertices. Then*

$$GSSD_f(G) \geq 2m.$$

Moreover, equality holds if and only if $f(u) = f(v)$, for all $uv \in E(G)$.

Proof. It is straightforward to observe that for any real positive number x, $x + \frac{1}{x} \geq 2$. Therefore, for any edge $uv \in E(G)$, we have:

$$\frac{f(u)}{f(v)} + \frac{f(v)}{f(u)} \geq 2.$$

Summing over all edges $uv \in E(G)$, we obtain:

$$GSSD_f(G) = \sum_{uv \in E(G)} \left(\frac{f(u)}{f(v)} + \frac{f(v)}{f(u)} \right) \geq 2m.$$

To prove the second statement, assume that $GSSD_f(G) = 2m$. Then, for each edge $uv \in E(G)$, we must have $\frac{f(u)}{f(v)} + \frac{f(v)}{f(u)} = 2$. This implies that $f(u) = f(v)$ for all $uv \in E(G)$.

Conversely, suppose that $f(u) = f(v)$ for all $uv \in E(G)$. Then, for each edge $uv \in E(G)$, we have $\frac{f(u)}{f(v)} + \frac{f(v)}{f(u)} = 2$. Therefore, $GSSD_f(G) = 2m$. Hence, if $f(u) = f(v)$ for all $uv \in E(G)$, then $SD_f(G) = 2m$. □

Finding the upper bound in the general case is not an easy task due to the unlimited functions. But it is possible to find upper bounds for this index in special cases and for specific functions.

Let f be a function on the vertices of a graph G, then for an edge $e = xy$ we define the edge f-orbit of e as $E_f(e) = \{uv \in E(G) | f(u) = f(x), f(v) = f(y)\}$. A graph G is said to be f-edge transitive if for any two given edges $e, e' \in E(G)$, $E_f(e) = E_f(e')$. Similarly, G can be deemed f-vertex transitive through a parallel definition.

Theorem 2.2. *Let G be a graph on n vertices, f be a function on the vertices of G, and $E_f(e_1), E_f(e_2), \ldots, E_f(e_k)$ be all edge f-orbits of G and $e_i = u_i v_i$ be an arbitrary edge of $E_f(e_i), (1 \leq i \leq k)$. Then*

$$GSSD_f(G) = \sum_{i=1}^{K} |E_f(e_i)| \frac{f^2(u_i) + f^2(v_i)}{f(u_i) f(v_i)}.$$

Proof. Since for two edges $e = u_i v_i$ and $e' = u'_i v'_i \in E_f(e_i)$, we have $\frac{f(u_i)}{f(v_i)} + \frac{f(v_i)}{f(u_i)} = \frac{f(u'_i)}{f(v'_i)} + \frac{f(v'_i)}{f(u'_i)}$, so we have the desired result. □

Corollary 2.3. *For a function f, let G be an f-edge-transitive graph on n vertices and m edges. Then*

$$GSSD_f(G) = m \left(\frac{\alpha^2 + \beta^2}{\alpha\beta} \right),$$

where for the arbitrary edge $e = uv$, we have $\alpha = f(u)$ and $\beta = f(v)$.

It is worth noting that in Equation (1) when $f(u) = d_u$, we obtain the same value as the SDD index. By replacing $f(u) = D(u)$ in place of $f(u)$ in Equation (1), we obtain the special case of $GSDD$ as follow:

$$GSDD_D(G) = \sum_{uv \in E(G)} \frac{D(u)}{D(v)} + \frac{D(v)}{D(u)}.$$

By substituting $f(u) = Sd(u)$ the following index concluded:

$$GSDD_{sd}(G) = \sum_{uv \in E(G)} \frac{sd(u)}{sd(v)} + \frac{sd(v)}{sd(u)}.$$

In the following, we calculate the value of $GSDD$ index of these specific functions for some well-known graphs.

Example 2.4. *Consider the complete bipartite graph $K_{m,n}$, with vertices $u_1, u_2, \ldots, u_m$ in one part and $v_1, v_2, \ldots, v_n$ in another part. for any edge $e = u_i v_j \in E(K_{m,n})$, we obtain*

$$\frac{D(u_i)}{D(v_j)} + \frac{D(v_j)}{D(u_i)} = \frac{2n+m-2}{2m+n-2} + \frac{2m+n-2}{2n+m-2}.$$

It is easy to see that $K_{m,n}$ is edge-transitive. So from Corollary 2.3 we have that

$$\begin{aligned} GSDD_D(K_{m,n}) &= \sum_{uv \in E(G)} \frac{D(u)}{D(v)} + \frac{D(v)}{D(u)} &(2)\\ &= mn \left(\frac{2n+m-2}{2m+n-2} + \frac{2m+n-2}{2n+m-2} \right) &(3)\\ &= \frac{mn(5m^2 + 8mn - 12m + 5n^2 - 12n + 8)}{(2m+n-2)(m+2n-2)}. &(4) \end{aligned}$$

On the other hand, for all $u \in V(G)$, we have $Sd(u) = mn$ and thus

$$GSDD_{Sd}(K_{m,n}) = \sum_{u \sim v} \left(\frac{Sd(u)}{Sd(v)} + \frac{Sd(v)}{Sd(u)} \right) = 2mn.$$

Example 2.5. *Recall that the friendship graph (or windmill graph or n-fan graph) F_n is a planar graph constructed from the union of n copies of K_2 and one copy of K_1 by joining the isolated vertex to all vertices of of degree one (see Figure 1.). It may be also pictured as a collection of n triangles with a common vertex. Let u be the central vertex of F_n, then $D(u) = 2n$ and for the non-central vertex v we have $D(v) = 4n - 2$. Hence*

$$GSDD_D(F_n) = \sum_{uv \in E(F_n)} \frac{D(u)}{D(v)} + \frac{D(v)}{D(u)} = 2n \left(\frac{2n}{4n-2} + \frac{4n-2}{2n} \right) + 2n = \frac{14n^2 - 10n + 2}{2n-1}.$$

We have also that $Sd(u) = 4n$ for the central vertex u, and $Sd(v) = 2n + 2$ for other vertices of F_n. This mean that

$$GSDD_{Sd}(F_n) = \sum_{uv \in E(F_n)} \frac{Sd(u)}{Sd(v)} + \frac{Sd(v)}{Sd(u)} = 2n \left(\frac{4n}{2n+2} + \frac{2n+2}{4n} \right) + 2n = \frac{7n^2 + 4n + 1}{n+1}.$$

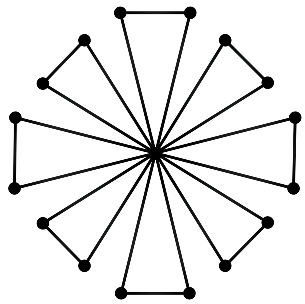

Figure 1. The friendship graph F_n with $n = 8$.

Example 2.6. *A wheel graph W_n is a graph formed by connecting a single vertex to all vertices of a cycle. For the central vertex u, we have that $D(u) = n - 1$ and for the non-central vertex v we have that $D(v) = 2n - 5$. Therefore*

$$GSDD_D(W_n) = \sum_{uv \in E(W_n)} \frac{D(u)}{D(v)} + \frac{D(v)}{D(u)} = (n-1)\left(\frac{n-1}{2n-5} + \frac{2n-5}{n-1}\right) + 2(n-1) = \frac{9(n-2)^2}{2n-5}.$$

For the $GSDD_{Sd}(W_n)$, the sum of the degrees of all the neighbors of the central vrtex is $3(n-1)$ and for orther vertices is $n+5$. Therefore

$$GSDD_{Sd}(W_n) = \sum_{uv \in E(W_n)} \frac{Sd(u)}{Sd(v)} + \frac{Sd(v)}{Sd(u)} = (n-1)\left(\frac{3(n-1)}{n+5} + \frac{n+5}{3(n-1)}\right) + 2(n-1)$$
$$= \frac{4(2n+1)^2}{3(n+5)}.$$

Example 2.7. *The double star graph $S_{m,n}$ is formed by connecting the centers of two stars $K_{1,m}$ and $K_{1,n}$ with an edge. Here, we compute the general symmetric degree distance (GSDD) of $S_{m,n}$ with respect to two different vertex functions:*

For $f(u) = D(u)$, we first note that the distance between any two vertices in $S_{m,n}$ is at most 2. We can compute $D(u)$ for each vertex u as follows:

- *For m pendant of $K_{1,m}$, we have $D(u) = 2m + 3n + 1$.*
- *For n pendant of $K_{1,n}$, we conclude $D(v) = 3m + 2n + 1$.*
- *For the two central vertices, we have $D(u) = m + 2n + 1$ and $D(v) = 2m + n + 1$.*

It may conclude that

$$\begin{aligned}
GSDD_D(S_{m,n}) &= \sum_{uv\in E(S_{m,n})}\left(\frac{D(u)}{D(v)}+\frac{D(v)}{D(u)}\right)\\
&= m\left(\frac{2m+3n+1}{m+2n+1}+\frac{m+2n+1}{2m+3n+1}\right)\\
&+ n\left(\frac{3m+2n+1}{2m+n+1}+\frac{2m+n+1}{3m+2n+1}\right)\\
&+ \left(\frac{2m+n+1}{m+2n+1}+\frac{m+2n+1}{2m+n+1}\right)\\
&= \frac{(18m+18n+4)(m+n+1)}{2(m+n)^2+7(m+n)+2}\\
&+ \frac{(2m+n+1)(m+2n+1)(2m+2n+2)}{(5(m+n)+1)(2(m+n)+1)}.
\end{aligned}$$

Similarly, for $f(u)=Sd(u)$, we have:

$$\begin{aligned}
GSDD_{Sd}(S_{m,n}) = \sum_{u\sim v}\left(\frac{Sd(u)}{Sd(v)}+\frac{Sd(v)}{Sd(u)}\right) &= m\left(\frac{m+1}{m+n+1}+\frac{m+n+1}{m+1}\right)\\
+ n\left(\frac{n+1}{m+n+1}+\frac{m+n+1}{n+1}\right)&+2.
\end{aligned}$$

3 Generalized symmetric division degree matrix

Following Rather et al. in [12], the general adjacency matrix (A_f-matrix) associated with the topological index of G is a real symmetric matrix, defined by

$$A_f(G)=(a_f)_{ij}=\begin{cases} f(u,v) & uv\in E(G)\\ 0 & \text{otherwise}\end{cases}.$$

The set of all eigenvalues of $A_f(G)$ is known as the general adjacency spectrum (A_f-spectrum) of G and are denoted by $\lambda_1(A_f(G))\geq\lambda_2(A_f(G))\geq\ldots\lambda_n(A_f(G))$, where $\lambda_1(A_f(G))$ is the general adjacency spectral radius. By Perron-Frobenius theorem it can be proved that $\lambda_1(A_f(G))$ is unique and its associated eigenvector has positive components.

Also, the generalized-energy of G is defined as

$$\mathcal{E}_f(G)=\sum_{i=1}^{n}|\lambda_i(A_f(G))|.$$

It is clear that if for each edge $e=uv, f(u)=f(v)$, then $A_f(G)=2A(G)$ and hence $spec_f(G)=2spec(G)$.

Example 3.1. *Suppose the graph G is vertex transitive.It is not difficult to prove that for any pair of vertices $u,v\in V(G)$, we have $D(u)=D(v)$. This mean that $A_D(G)=2A(G)$ and $spec_D(G)=$*

$2spec(G)$. *Also, since the graph G is k-regular, for each vertex u,* $Sd(u) = k^2$ *and thus* $A_{Sd}(G) = 2A(G)$, *which yields that* $spec_{Sd}(G) = 2spec(G)$.

Example 3.2. *Suppose G is an f-edge-transitive graph. By Corollary 2.3, we conclude that* $A_f(G) = \left(\frac{f(u)}{f(v)} + \frac{f(v)}{f(u)}\right) A(G)$, *for each edge* $e = uv$. *Hence* $\lambda_i(A_f(G)) = \left(\frac{\alpha^2+1}{\alpha}\right) \lambda_i(A(G))$, *where* $\alpha = \frac{f(u)}{f(v)}$. *So we have the following theorem without proof.*

Theorem 3.3.

1. *Suppose G is an f-edge-transitive graph. Then*
$$\mathcal{E}_f(G) = \frac{\alpha^2+1}{\alpha}\mathcal{E}(G),$$
where $\alpha = \frac{f(u)}{f(v)}$ *for an arbitrary edge* $e = uv \in E(G)$.
2. *If G is vertex-transitive graph, then*
$$\mathcal{E}_D(G) = 2\mathcal{E}(G) \text{ and } \mathcal{E}_{Sd}(G) = 2\mathcal{E}(G).$$
3. *In general* $\mathcal{E}_f(G) \geq 2\mathcal{E}(G)$.

Definition 1. Suppose G is a connected graph and f is a positive real-valued function on vertices, where $\sum_{uv\in E(G)} f(u,v)$ is a topological index. Then the f-laplacian matrix is defined as $\mathcal{L}_f = D_f(G) - A_f(G)$, where $D_f(G)$ is a diagonal matrix with diagonal entries $(D_f)_i = f(u)$.

The eigenvalues of$\mathcal{L}_f$ are called f-laplacian eigenvalues of G denoted by $\mu_1(D_f(G)) \leq \mu_2(D_f(G)) \leq \cdots \leq \mu_n(D_f(G))$. Regarding the ordinary laplacian energy, the f-laplacian energy is defined as
$$\mathcal{LE}_f(G) = \sum_{i=1}^{n} |\mu_i - \frac{\sum_{u\in V(G)} f(u)}{n}|.$$
It is clear that if $f(u) = deg(u)$, then $\mathcal{LE}_f(G) = \mathcal{LE}(G)$, since $\sum_{u\in V(G)} f(u) = \sum_{u\in V(G)} deg(u) = 2m$.

Example 3.4. *If G is a connected graph in which* $f(u) = D(u)$, *then* $\sum_{u\in V(G)} f(u) = \sum_{u\in V(G)} D(u) = 2W(G)$, *and thus*
$$\mathcal{LE}_f(G) = \sum_{i=1}^{n} |\mu_i - \frac{2W(G)}{n}|.$$

If further G is vertex-transitive, it is not difficult to see that for each vertex $u \in V(G)$, $W(G) = \frac{n}{2}D(u)$, *and hence,* $\frac{2W(G)}{n} = D(u)$. *Thus*
$$\mathcal{LE}_f(G) = \sum_{i=1}^{n} |\mu_i - D(u)|,$$
for a vertex $u \in V(G)$.

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