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Research Paper

Power graphs via their characteristic polynomial

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Abstract. A power graph is defined a graph that it's vertices are the elements of group and two vertices are adjacent if and only if one of them is a power of the other. Suppose $A(X)$ is the adjacency matrix of graph X. Then the polynomial $\chi(X,\lambda) = det(xI - A(X))$ is called as characteristic polynomial of X. In this paper, we compute the characteristic polynomial of all power graphs of order p^2q, where p, q are distinct prime numbers.

Keywords: power graph, characteristic polynomial, generelized coalescence
Mathematics Subject Classification (2010): $05C10$, $05C25$, $20B25$.

1 Introduction

Kelarve and Quinn in [13] introduced the directed power graph of a semi-group. The undirected power graph $\mathcal{P}(S)$ of a semigroup S is defined by Chakrabarty et al in which the set of vertices is the elements of S and two distinct vertices are adjacent if and only if one of them is a power of the other, see [5]. They proved that $\mathcal{P}(G)$ is a complete graph if and only if G is a cyclic group of order p^m, where p is a prime number and m is a positive integer and also, they obtained a formula for the number of edges in a finite power graph. Cameron and Gosh [3] proved non-isomorphic abelian groups don't have isomorphic power graphs, but non-abelian groups may have this condition. Ghorbani et al. in [9] determined the structure of power graphs of all groups of order a product of three distinct prime numbers. By continuing

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this method, here we determine the characteristic polynomial of power graphs of groups of order pq and p^2q, where p,q are distinct prime numbers. The polynomial $\chi(X,\lambda) = det(xI - A(X))$ is called as characteristic polynomial of graph X.

Let $x,y \in G$ be two arbitary elements such that there is an edge between them in $\mathcal{P}(G)$, then for the smallest positive integer r, we have $x^r = y$. Now, it is easy to see that $\{m \in \mathbf{N} : x^m = y\}$ is the arithmetic progression with initial term r and common difference $d = o(x)$ denoted by $AP(r,d)$. Let us to get $A(X)$ is the arc set of a graph X and $B = \{(v,v) : v \in V(X)\}$. We mean a function by a generalization on X as $W : A(X) \cup B \to \mathbf{N} \cup \{0\} \times \mathbf{N} \cup \{0\}$.

2 Definitions and Preliminaries

Let (X_1, W_1) and (X_2, W_2) be to graphs equipped with two generalizations W_1, W_2 respectively. Then the generalized product $X_1 \times_W X_2$ is a graph with vertex set $V(X_1) \times V(X_2)$ and $(g_1, g_2) \sim (g_1', g_2')$ if and only if the following two conditions hold simultaneously:
(i) $(g_1, g_2) \neq (g_1', g_2')$ and
(ii) $AP(W_1(g_1, g_1')) \cap AP(W_2(g_2, g_2')) \cap \mathbf{N} \neq \emptyset$ or
$AP(W_1(g_1', g_1)) \cap AP(W_2(g_2', g_2)) \cap \mathbf{N} \neq \emptyset$,
see [2] for more details.

Theorem 2.1. *[2] Let G be a finite group. Then $\mathcal{P}(G)$ is complete graph if and only if G is a cyclic group of order 1 or p^m, for some prime number p and $m \in \mathbf{N}$.*

Theorem 2.2. *[2] For two groups G_1 and G_2, $\mathcal{P}(G_1 \times G_2)$ and $\mathcal{P}(G_1) \times_W \mathcal{P}(G_2)$ are isomorphic for some choice of generalizations W_1 and W_2 of $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$ respectively.*

Theorem 2.3. *[7] The characteristic polynomial of the disjoint union of two graphs X_1 and X_2 is*

$$\chi(X_1 \cup X_2, \lambda) = \chi(X_1, \lambda)\chi(X_2, \lambda).$$

Theorem 2.3 yields that if $X_1, X_2, \ldots, X_s$ are the components of the graph X, then

$$\chi(X, \lambda) = \chi(X_1, \lambda)\chi(X_2, \lambda)\ldots\chi(X_s, \lambda).$$

Suppose $X = X_1 + X_2$ is the join graph of X_1 and X_2 with vertex set $V(X) = \cup_{i=1}^{2} V(X_i)$ and edge set

$$E(X) = \cup_{i=1}^{2} E(X_i) \cup \{(u,v) | u \in V(X_i), v \in V(X_j), \ (1 \leq i,j \leq 2)\}.$$

Then, we have the following theorem.

Theorem 2.4. *[7] Let X_1, X_2 be two graphs on respectively n_1, n_2 vertices. The characteristic polynomial of $X_1 + X_2$ is*

$$\begin{aligned}\chi(X_1 + X_2, \lambda) = (-1)^{n_2}\chi(X_1, \lambda)\chi(\bar{X}_2, -\lambda - 1) + (-1)^{n_1}\chi(X_2, \lambda)\chi(\bar{X}_1, -\lambda - 1) \\ - (-1)^{n_1+n_2}\chi(\bar{X}_1, -\lambda - 1)\chi(\bar{X}_2, -\lambda - 1).\end{aligned}$$

Suppose the numbers $\beta_i = \frac{||P_i j||}{\sqrt{n}}$, $(i = 1, \ldots, m)$ are the main angles of graph Γ; they are the cosines of the angles between eigenspaces and j, see [6]. Note that $\Sigma_{i=1}^{m} \beta_i^2 = 1$, because $\Sigma_{i=1}^{m} P_i j = j$. Also, suppose μ_i are the distinct eigenvalues of X. Then we have the following proposition.

Proposition 2.5. *[7] For given graph X, we have*

$$\chi(K_1 + X, \lambda) = \chi(X, \lambda)(\lambda - \Sigma_{i=1}^{m} \frac{n\beta_i^2}{\lambda - \mu_i}).$$

Theorem 2.6. *[6] The characteristic polynomial of the power graph of the cyclic group $\mathbb{Z}_n$ is*

$$\chi(\mathcal{P}(\mathbb{Z}_n), \lambda) = \chi(T, \lambda)(\lambda + 1)^{n-t-1},$$

where d_i's $(1 \leq i \leq t)$, are all non-trivial divisors of n,

$$T = \begin{pmatrix} \varphi(n) & \varphi(d_1) & \varphi(d_2) & \ldots & \varphi(d_t) \\ \varphi(n)+1 & \varphi(d_1)-1 & \alpha_{d_1 d_2} & \ldots & \alpha_{d_1 d_t} \\ \varphi(n)+1 & \alpha_{d_2 d_1} & \varphi(d_2)-1 & \ldots & \alpha_{d_2 d_t} \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ \varphi(n)+1 & \alpha_{d_t d_1} & \alpha_{d_t d_2} & \ldots & \varphi(d_t)-1 \end{pmatrix}$$

and

$$\alpha_{d_i d_j} = \begin{cases} \varphi(d_j) & d_i \mid d_j \text{ or } d_j \mid d_i \\ 0 & \text{otherwise} \end{cases}.$$

The coalescence graph $X_1.X_2$ of two graphs X_1 and X_2 obtained from disjoint union $X_1 \cup X_2$ by identifying a vertex u of X_1 with a vertex v of X_2. In [6] it is proved that

$$\chi(X_1.X_2, \lambda) = \chi(X_1, \lambda)\chi(X_2 - v, \lambda) + \chi(X_1 - u, \lambda)\chi(X_2, \lambda) - \lambda\chi(X_1 - u, \lambda)\chi(X_2 - v, \lambda).$$

Now, suppose X_1, X_2 have respectively subgraphs S, S' where $S \cong S'$ and suppose $X_1(X_2)$ has a vertex $u(v)$ adjacent to all vertices of $S(S')$. We can define the generelized coalescence $X_1 * X_2$ of two graphs X_1, X_2 by identifying the vertices of subgraph S with the vertices of subgraph S'.

Theorem 2.7. *[9] The characteristic polynomial of generelized coalescence $X_1 * X_2$ is*

$$\begin{aligned}\chi(X_1 * X_2, \lambda) &= \chi(X_1, \lambda)\chi(X_2 - S, \lambda) + \chi(X_1 - S, \lambda)\chi(X_2, \lambda) \\ &- \chi(S, \lambda)\chi(X_1 - S, \lambda)\chi(X_2 - S, \lambda).\end{aligned}$$

3 Main Results

It is well-known that up to isomorphism there are only two groups of order pq namely $\mathbb{Z}_{pq}$ and $F_{p,q}$ $(q|p-1)$. Suppose $\mathcal{G}(p^2, q)$ is the class of all groups of order p^2q, where p and q

are prime numbers. In [9,10] it is proved that a group of order p^2q is isomorphic with one of the following structures:

Case 1. $(p<q)$ $\mathbb{Z}_{p^2q}$, $\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_q$, $\mathbb{Z}_p\times F_{q,p}$ $(p|q-1)$, F_{q,p^2} $(p^2|q-1)$, $\langle a,b : a^{p^2}=b^q=1, a^{-1}ba=b^{\alpha},\ \alpha^p\equiv 1\ (mod\ q)\rangle$.

Case 2. $(q<p)$ $\mathbb{Z}_{p^2q}$, $\mathbb{Z}_p\times\mathbb{Z}_q\times\mathbb{Z}_p$, $\mathbb{Z}_p\times F_{p,q}$ $(q|p-1)$, $F_{p^2,q}$ $(q|p^2-1)$, $\langle a,b,c : a^p=b^q=c^p=1, ac=ca, b^{-1}ab=a^{\alpha}, b^{-1}cb=c^{\alpha^x},\ \alpha^q c, x=1,\ldots,q-1\rangle$, $\langle a,b,c : a^p=b^q=c^p=1, ac=ca, b^{-1}ab=a^{\alpha}c^{\beta D}, b^{-1}cb=a^{\beta}c^{\alpha}\rangle$, where $\alpha+\beta\sqrt{D}=\sigma^{p^2-1/q}$, σ is a primitive element of $GF(p^2)$, $q\nmid p-1$ and $q\neq 2$ whereas $q|p+1$. First, we recall that the number of generators of the abelian group $\mathbb{Z}_{pq}$ is $\varphi(pq)$. This indicate that there is a clique of order $\varphi(pq)$, where φ denotes the Euiler's function. The vertices of forms a^{ip} $(1\le i\le q-1)$ and a^{jq} $(1\le j\le p-1)$, where a is a generator of group yields two cliques of orders $q-1$ and $p-1$, respectively. By using the structure of an abelian group, all of them are distinct. The structure of power graph $\mathcal{P}(\mathbb{Z}_{pq})$ is depicted in Figure 1. It should be noted that in Figure 1, $K=K_{\varphi(pq)+1}$.

Theorem 3.1. *Suppose $G\cong\mathbb{Z}_{pq}=\langle a\rangle$. Then $\mathcal{P}(G)\cong K_{\varphi(pq)+1}+(K_{p-1}\cup K_{q-1})$.*

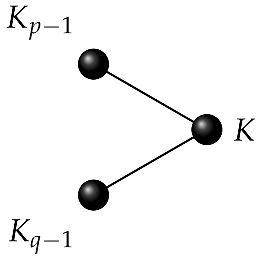

Figure 1. The structure of power graph $\mathcal{P}(\mathbb{Z}_{pq})$.

Corollary 3.2. *Let $\alpha=(p-1)(q-1)$. The characteristic polynomial of graph $\mathcal{P}(\mathbb{Z}_{pq})$ is*

$$\chi(X,\lambda)=\chi(T,\lambda)(\lambda+1)^{pq-3}$$

where

$$T=\begin{pmatrix}\alpha & q-1 & p-1\\ \alpha+1 & q-2 & 0\\ \alpha+1 & 0 & p-2\end{pmatrix}.$$

Proof. Use Theorem 1.5. □

Here, consider the Frobenius group $F_{p,q}$ by presentation $F_{p,q}=\langle a,b : a^p=b^q=1, b^{-1}ab=a^u\rangle$, where u is an element of order q in multiplicative group $\mathbb{Z}_p^*$. One can see the elements a^i's $(1\le i\le p-1)$ and b^j's $(1\le j\le q-1)$ respectively, introduce two cliques of orders $p-1$ and $q-1$. Consider the vertices b^ja^i $(1\le i\le p-1,\ 1\le j\le q-1)$, by the relation $b^{-1}ab=a^u$. We claim that

$$(b^ja^i)^m=b^{jm}a^{i(u^{j(m-1)}+\cdots+u^j+1)}.$$

Therefore, we can prove that $o(b^j a^i) = q$ that derive $p-1$ distinct cliques of order $q-1$. Assume these elements are adjacent with a^i's. Then one can see that there exist an integer $1 \le m \le q-1$ such that $(b^j a^i)^m = a^{i'}$ and so $q \mid jm$, a contradiction. By a similar way, we can conclude these vertices are distinct from b^j's. The related graph is depicted in Figure 2.

To do this, let $m = 1$, then $(y^j x^i)^1 = y^j x^{i(u^{j(1-1)})} = y^j x^i$. We have

$$\begin{aligned}(y^j x^i)^{m+1} &= (y^j x^i) * (y^j x^i)^m = (y^j x^i) * y^{jm} x^{i(u^{j(m-1)}+\cdots+u^j+1)} \\ &= y^{(m+1)j} y^{-jm} x^i y^{jm} x^{i(u^{j(m-1)}+\cdots+u^j+1)} \\ &= y^{(m+1)j} x^{i(u^{jm})} x^{i(u^{j(m-1)}+\cdots+u^j+1)} \\ &= y^{(m+1)j} x^{i(u^{jm}+\cdots+u^j+1)}.\end{aligned}$$

We summarize the above results in the following theorem.

Theorem 3.3. *Suppose $G \cong F_{p,q}$. Then $\mathcal{P}(G) \cong K_1 + (K_{p-1} \cup (\cup_{i=1}^{p} K_{q-1}))$. The structure of power graph $\mathcal{P}(F_{p,q})$ is given in Figure 2.*

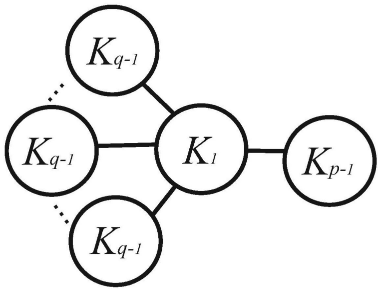

Figure 2. The structure of power graph $\mathcal{P}(F_{p,q})$.

Corollary 3.4. *The characteristic polynomial of graph $\mathcal{P}(F_{p,q})$ is*

$$\begin{aligned}\chi(X,\lambda) = (\lambda+1)^{p(q-1)-2}(\lambda-(q-2))^{p-1}(\lambda^3 - (p+q-4)\lambda^2 \\ -(2p+2q-5)\lambda + (p-1)^2(q-1) + (p-2)(q-2) - 1)\end{aligned}$$

Proof. Assume that $X' = (\cup_{i=1}^{p} K_{q-1}) \cup K_{p-1}$, then by using Theorem 2.2, we have $\chi(X',\lambda) = (\lambda+1)^{p(q-1)-2}(\lambda-(q-2))^p(\lambda-(p-2))$. On the ohert hand, by a simple method, we can see that $\bar{X}' = K_{q-1,\ldots,q-1,p-1}$. This implies that

$$\chi(\bar{X}',\lambda) = \lambda^{p(q-1)-2}(\lambda+q-1)^{p-1}(\lambda^2 - (p-1)(q-1)\lambda - p(p-1)(q-1)).$$

Now, apply Theorem 2.3 to complete the proof. □

3.1 The structure of $\mathcal{P}(G)$, where $|G| = p^2 q$ $(p < q)$

Suppose $X_1, \ldots, X_n$ are n connected graphs. The graph $P_n[X_1, \ldots, X_n]$ is a graph constructed by $\cup_{i=1}^{n} X_i$ in which every vertex of X_i is adjacent with every vertex of X_{i+1} for $1 \le i \le n-1$.

Theorem 3.5. *Suppose* $G \cong \mathbb{Z}_{p^2q} = \langle a \rangle$. *Then*

$$\mathcal{P}(G) \cong K_{\varphi(p^2q)+1} + P_4[K_{p^2-1}, K_{p-1}, K_{pq-1}, K_{q-1}].$$

Proof. For any non-trivial devisor d of p^2q, the abelian group $\mathbb{Z}_{p^2q}$ has a cyclic subgroup of order d. Therefore, the vertices of $\mathcal{P}(G)$ can be partitioned to five subsets. The elements a^{ipq} $(1 \leq i \leq p-1)$, a^{jq} $(1 \leq j \leq p^2-1)$, a^{kp^2} $(1 \leq k \leq q-1)$, a^{tp} $(1 \leq t \leq pq-1)$ and the generators of G. By using Theorem 2.1, we acheive five cliques of orders $p-1$, p^2-1, $q-1$, $pq-1$ and $\varphi(p^2q)$, respectively. Now by applying the following relations, we can describe adjacency between different cliques:

$\langle a^{ipq} \rangle \subseteq \langle a^{tp} \rangle, \langle a^{jq} \rangle$, $\langle a^{tp} \rangle \subseteq \langle a^{kp^2} \rangle$, $\langle a^{iq} \rangle \subseteq \langle a^{kp^2} \rangle$.

The structure of power graph $\mathcal{P}(G)$ is depicted in Figure 3. This completes the proof. □

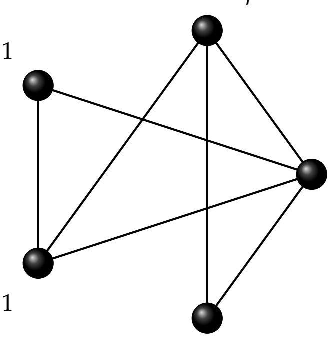

Figure 3. The structure of power graph $\mathcal{P}(\mathbb{Z}_{p^2q})$.

Corollary 3.6. *The characteristic polynomial of graph* $\mathcal{P}(\mathbb{Z}_{p^2q})$ *is*

$$\chi(X,\lambda) = \chi(T,\lambda)(\lambda+1)^{p^2q-5}$$

where

$$T = \begin{pmatrix} \alpha & p-1 & q-1 & \gamma & \beta \\ \alpha+1 & p-2 & 0 & \gamma & \beta \\ \alpha+1 & 0 & q-2 & 0 & \beta \\ \alpha+1 & p-1 & 0 & \gamma-1 & 0 \\ \alpha+1 & p-1 & q-1 & 0 & \beta-1 \end{pmatrix}$$

$\alpha = p(p-1)(q-1)$, $\beta = (p-1)(q-1)$ *and* $\gamma = p(p-1)$.

Proof. Use Theorem 2.4. □

Theorem 3.7. *Let*

$$G \cong \mathbb{Z}_p \times \mathbb{Z}_{pq} = \langle x, y : x^p = y^{pq} = 1, xy = yx \rangle.$$

Then $\mathcal{P}(G) = K_1 + (X_1 * X_2 * \cdots * X_{p+1})$, *where*

$$X_i = K_{pq-p-q+1} + (K_{p-1} \cup K_{q-1})\ (1 \le i \le p+1).$$

Proof. In the first step, we can consider the following generalization W_1 of $\mathcal{P}(G_1)$ as:

$$W_1(x,z) = \begin{cases} (r, o(x)) \ if\ r \text{ is the smallest positive integer such that } x^r = z \\ (0,0) \qquad\qquad\qquad\qquad otherwise \end{cases}$$

and the generalization W_2 of $\mathcal{P}(G_2)$, similarly. Then by using Theorem 1.2, we get our result. The structure of power praph of this group is as shown in Figure 4. □

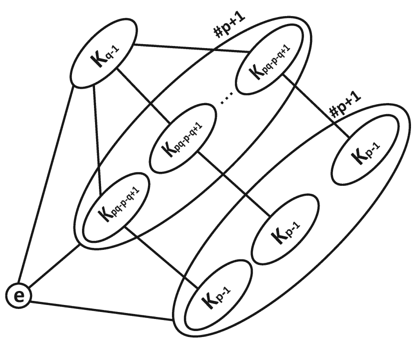

Figure 4. The structure of power graph $\mathcal{P}(\mathbb{Z}_p \times \mathbb{Z}_{pq})$.

Corollary 3.8. *By above notation the characteristic polynomial of graph* $X = X_1 * X_2 * \cdots * X_{p+1}$ *is*

$$\begin{aligned}\chi(X,\lambda) = {} & (\lambda+1)^{p(pq-1)-4}(\lambda-(pq-q-1))^p(\lambda^3-(pq-4)\lambda^2 \\ & - (pq((p-1)(q-2)+1)+p^2+q-6)\lambda \\ & + (p+1)((p-1)^2(q-1)^2-(p+q-3)) \\ & - p(q-2)(pq-q-1)).\end{aligned}$$

Proof. Assume that $X_i = K_{pq-p-q+1} + (K_{p-1} \cup K_{q-1})$ $(1 \le i \le p+1)$, then by using Theorem 2.3, we have

$$\begin{aligned}\chi(X_i,\lambda) = {} & (\lambda+1)^{pq-4}(\lambda^3-(pq-4)\lambda^2-((p+1)(q+1)-7)\lambda \\ & + (p-1)(q-1)(pq-p-q)+(p-2)(q-2)).\end{aligned}$$

The characteristic polynomial of $K_1 + (X_1 * X_2 * \cdots * X_{p+1})$ follows immediately from the Theorem 2.5 and Proposition 2.1. □

Theorem 3.9. *Let*

$$G \cong \mathbb{Z}_p \times F_{q,p}\ (p|q-1) = \langle a,b,c : a^p = b^q = c^p = 1, c^{-1}bc = b^u \rangle,$$

where $u^p \equiv 1\ (mod\ q)$. *Then*

$$\mathcal{P}(G) = K_1 + ((K_{pq-p-q+1} + (K_{p-1} \cup K_{q-1})) \cup (\cup_{i=1}^{pq} K_{p-1})).$$

Proof. The proof is similarto that of Theorem 2.4. The structure of these power garph is depicted in Figure 5. □

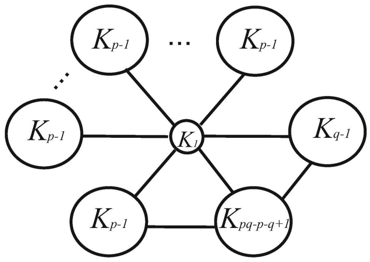

Figure 5. The power graph $\mathcal{P}(\mathbb{Z}_p \times F_{q,p})$.

Corollary 3.10. *The characteristic polynomial of graph* $X = \mathcal{P}(\mathbb{Z}_p \times F_{q,p})$ *is*

$$\chi(X,\lambda) = (\lambda+1)^{pq(p-1)-4}(\lambda-(p-2))^{pq-1}(\lambda^5 - (pq+p-6)\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta,$$

where $\alpha = (2pq+q-3)(p-1) - 4p(q+1) + 14$, $\beta = (p-1)^2(q-1)^2 + (p-3)(pq+p+q-6) + (p-2)(pq+q-6) - (p-1)(p^2q^2 - 7pq + 11) - 2q$ *and* $\gamma = (p-1)(pq-1)(-2pq+7) + (p-1)(q-1)(3(p-1)(q-2)-1) + (p-2)(3p+4q-12)$, $\delta = (p-1)^2(pq-6) + (p-1)(q-1)(-p((pq-p-q)^2 + 2pq) - 5) + 3pq^2(p-1) - 1)$.

Proof. In view of Theorem 1.4, it is sufficent to consider that $X_1 \cong K_{pq-p-q+1} + (K_{p-1} \cup K_{q-1})$ and $X_2 \cong X_1 \cup (\cup_{i=1}^{pq} K_{p-1})$, then

$$\chi(X_1,\lambda) = (\lambda+1)^{pq-4}(\lambda^3 - (pq-4)\lambda^2 - ((p+1)(q+1)-7)\lambda + (p-1)(q-1)(pq-p-q) + (p-2)(q-2)).$$

Hence, Therorem 2.2 yields

$$\chi(X_2,\lambda) = (\lambda+1)^{pq(p-1)-4}(\lambda-p+2)^{pq}(\lambda^3 - (pq-4)\lambda^2 - (pq+p+q-6)\lambda + (p-1)(q-1)(pq-p-q) + (p-2)(q-2)).$$

On the other hand, the structure of X_2 implies that

$$\bar{X}_2 \cong (\bar{K}_{pq-p-q+1} \cup K_{p-1,q-1}) + K_{p-1,\ldots,p-1}$$

and thus

$$\begin{aligned}\chi(\bar{X}_2,\lambda) = \lambda^{pq(p-1)-4}(\lambda + p - 1)^{pq-1}(\lambda^4 + (pq-1)(p-1)\lambda^3 \\ + ((p-1)((pq-1)(pq-2) - q + 1))\lambda^2 \\ + ((p-1)^2(q-1)(pq-3))\lambda \\ + (p-1)(pq-2)((p-1)^2(q-1)^2 - 2p - 2q + 6)).\end{aligned}$$

□

Suppose $G \cong F_{q,p^2}$ $(p^2|q-1) = \langle x,y : x^q = y^{p^2} = 1, y^{-1}xy = x^u\rangle$, then by the presentation of the group G and by a similar argument, we can conclude the following theorem.

Theorem 3.11. *Suppose $G \cong F_{q,p^2}$ $(p^2|q-1) = \langle x,y : x^q = y^{p^2} = 1, y^{-1}xy = x^u\rangle$, where $u^{p^2} \equiv 1$ $(mod\ q)$. Then*

$$\mathcal{P}(G) = K_1 + ((\cup_{i=1}^{q} K_{p^2-1}) \cup K_{q-1}).$$

We can see the structure of it's power garph is as given in Figure 6.

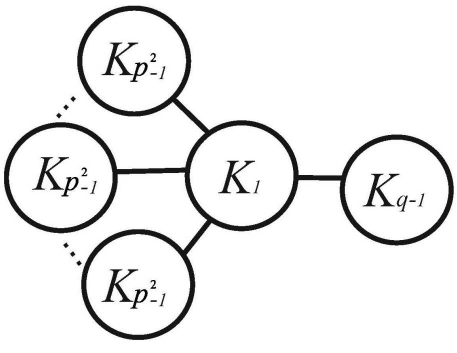

Figure 6. The power graph $\mathcal{P}(F_{q,p^2})$.

Corollary 3.12. *The characteristic polynomial of graph $X = \mathcal{P}(F_{q,p^2})$ is*

$$\begin{aligned}\chi(X,\lambda) = (\lambda+1)^{q(p^2-1)-2}(\lambda - (p^2-2))^{q-1}(\lambda^3 - (p^2+q-4)\lambda^2 \\ - (2p^2 + 2q - 5)\lambda + q^2(p^2-1) - p^2(q+1) + 2)\end{aligned}$$

Proof. Assume that $X' = (\cup_{i=1}^{q} K_{p^2-1}) \cup K_{q-1}$, then by using Theorem 1.3, we have $\chi(X',\lambda) = (\lambda+1)^{q(p^2-1)-2}(\lambda - (p^2-2))^q(\lambda - (q-2))$. On the ohert hand, by a simple method, we can see that $\bar{X}' = K_{p^2-1,\dots,p^2-1,q-1}$. This implies that

$$\chi(\bar{X}',\lambda) = (\lambda)^{q(p^2-1)-2}(\lambda + p^2 - 1)^{q-1}(\lambda^2 - (q-1)(p^2-1)\lambda - q(q-1)(p^2-1)).$$

Now apply Theorem 1.4 to complete the proof. □

Theorem 3.13. *Let $G \cong \langle a,b : a^{p^2} = b^q = 1, a^{-1}ba = b^\alpha\rangle$, where $\alpha^p \equiv 1$ $(mod\ q)$. Then $\mathcal{P}(G) = K_1 + (((\cup_{i=1}^{q} K_{p^2-p}) + K_{p-1}) * (K_{pq-p-q+1} + (K_{p-1} \cup K_{q-1})))$.*

Proof. In this group, the elements a^i $(1 \leq i \leq p^2-1)$, b^j $(1 \leq j \leq q-1)$ compose two cliques of orders p^2-1 and $q-1$, respectively. The elements a^ib^j $(1 \leq i \leq p^2-1)$, $(1 \leq j \leq q-1)$ satisfy in relation $(a^ib^j)^m = a^{im}b^{j(\alpha^{i(m-1)}+\cdots+\alpha^i+1)}$ and we can consider two following cases:

Case 1. Assume $i \neq kp$, then $o(a^ib^j) = p^2$ yields $q-1$ cliques of order p^2-p. We can prove that $(a^ib^j)^{lp} = a^{ilp}$ $(1 \leq l \leq p-1)$ that implies these vertices are adjacent with the elements a^i $(i = kp)$'s.

Case 2. If $i = kq$, then $o(a^ib^j) = pq$, $(a^ib^j)^{lp} = b^m$ $(1 \leq m \leq q-1)$, $(1 \leq l \leq p-1)$ and $(a^ib^j)^{tq} = a^{itp}$. Therefore, we acheive a clique of order $pq-p-q+1$ in which their vertices are adjacent with the elements a^i's $(i = kq)$ and b^j's. The structure of power graph $\mathcal{P}(G)$ is depicted in Figure 7. □

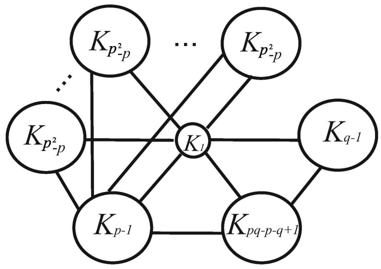

Figure 7. The structure of power graph $\mathcal{P}(G)$.

Corollary 3.14. *The characteristic polynomial of graph*

$$X = \mathcal{P}(\langle a,b : a^{p^2} = b^q = 1, a^{-1}ba = b^\alpha \rangle)$$

is

$$\chi(X,\lambda) = (\lambda+1)^{q(p^2-1)-4}(\lambda-(p^2-p-1))^{q-1}(\lambda^4-(p(p+q-1)-5)\lambda^3 + ((p(p-3)-1)(q-1)-3(p^2-3))\lambda^2+\alpha\lambda+\beta$$

where $\alpha = (q-1)(q(p-1)^2-2)+pq(p-1)(pq(p-1)-p^2+2)+(p-1)(p^2-5p-3)+5$ *and* $\beta = pq(p-1)^2(pq-p-1)-(p^2-p-1)(p-1)^2(q-1)^2+(p^2-p-1)(p+q-3)$.

Proof. Assume $X_1 \cong (\cup_{i=1}^q K_{p^2-p}) + K_{p-1}$ and $X_2 \cong K_{pq-p-q+1} + (K_{p-1} \cup K_{q-1})$, then

$$\chi(X_1,\lambda) = (\lambda+1)^{q(p^2-p-1)+p-2}(\lambda-(p^2-p-1))^{q-1}(\lambda^2 - (p^2-3)\lambda-(p^2-2)-p(p-1)^2(q-1))$$

and

$$\chi(X_2,\lambda) = (\lambda+1)^{pq-4}(\lambda^3-(pq-4)\lambda^2-(pq+p+q-6)\lambda + (p-1)(q-1)(pq-p-q)+(p-2)(q-2)).$$

On the other hand, $X_1 - K_{p-1} = \cup_{i=1}^q K_{p^2-p}$, $X_2 - K_{p-1} = K_{pq-p}$ and by Theorem 2.5 the proof is complete. □

3.2 The power graphs of groups of order p^2q where $p > q$

In this section, we apply a similar methods given in the last section to determine the structure of $\mathcal{P}(G)$, where G is isomorphic to a finite group of order p^2q, where $p > q$. The power graphs of groups $\mathbb{Z}_{p^2q}$, $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_p$, $\mathbb{Z}_p \times F_{p,q}$ and $F_{p^2,q}$ are given in Theorems 3.1-3.3. In what follows, we explain how we compute the power graphs of the other groups of this order.

Theorem 3.15. *Suppose*

$$G \cong \langle a,b,c : a^p = b^q = c^p = 1, ac = ca, b^{-1}ab = a^{\alpha}, b^{-1}cb = c^{\alpha^x} \rangle,$$

where $\alpha^q \equiv 1 \ (mod\ p)$, $x = 1,\ldots,q-1$. *Then*

$$\mathcal{P}(G) = K_1 + ((\cup_{i=1}^{p+1} K_{p-1}) \cup (\cup_{i=1}^{p^2} K_{q-1})).$$

Proof. The vertices corresponded to the elements a^i's $(1 \le i \le p-1)$, b^j's $(1 \le j \le q-1)$ and c^k's $(1 \le k \le p-1)$ compose three cliques of order respectively, $p-1$, $q-1$ and $p-1$. For elements b^ja^i's $(1 \le i \le p-1,\ 1 \le j \le q-1)$, by using the relation $b^{-1}ab = a^{\alpha}$ and $(b^ja^i)^m = b^{jm}a^{i(\alpha^{j(m-1)}+\cdots+\alpha^j+1)}$, we obtain $o(b^ja^i) = q$ which yields $p-1$ cliques of order $q-1$. Consider now the elements a^ic^k's $(1 \le i,k \le p-1)$. The relation $ac = ca$ yields $o(a^ic^k) = p$ and then we acheive $p-1$ cliques of order $p-1$. By the structure of group G, the elements b^jc^k's $(1 \le j \le q-1,\ 1 \le k \le p-1)$ form $p-1$ cliques of order $q-1$ and the relation $b^{-1}cb = c^{\alpha^x}$ verify that these vertices are distinct from other elements. The elements $c^kb^ja^i$ $(1 \le i,k \le p-1,\ 1 \le j \le q-1)$ are of order q, hence by using induction we get that

$$(c^kb^ja^i)^m = c^{km}b^{jm}a^{i(u^{j(m-1)}+\cdots+u^j+1)}.$$

Thus, we have $(p-1)^2$ new cliques of order $q-1$. Also, the relations of group yield these vertices are distinct from the other vertices. The structure of power graph of G is depicted in Figure 8. □

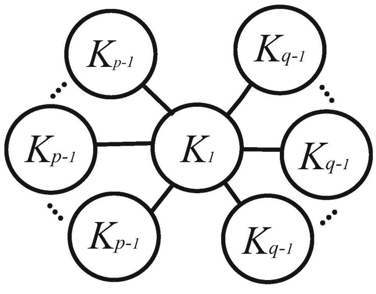

Figure 8. The structure of power graph $\mathcal{P}(G)$.

Corollary 3.16. *The characteristic polynomial of graph*

$$X = \mathcal{P}(\langle a,b,c : a^p = b^q = c^p = 1, ac = ca, b^{-1}ab = a^{\alpha}, b^{-1}cb = c^{\alpha^x} \rangle)$$

is

$$\begin{aligned}\chi(X,\lambda) = (\lambda+1)^{p^2(q-1)-p-2}(\lambda-(p-2))^p(x-(q-2))^{p^2-1}(\lambda^3-(p+q-4)\lambda^2 \\ -(p(p^2-1)-(p-2)(q-2)+q-1)\lambda+(p^2-1)((p-1)^2+q-1) \\ +(p-2)(q-2)-p^3+2p-2)\end{aligned}$$

Proof. First apply Theorem 1.3, to compute the characteristic polynomial of $Y \cong (\cup_{i=1}^{p+1} K_{p-1}) \cup (\cup_{i=1}^{p^2} K_{q-1})$ as follows

$$\chi(Y,\lambda) = (\lambda+1)^{p^2(q-1)-p-2}(\lambda-(p-2))^{p+1}(\lambda-(q-2))^{p^2}.$$

Also, we can see $\bar{Y} = K_{p-1,\ldots,p-1} + K_{q-1,\ldots,q-1}$ and

$$\begin{aligned}\chi(\bar{Y},\lambda) = \lambda^{p^2(q-1)-p-2}(\lambda+p-1)^p(\lambda+q-1)^{p^2-1}(\lambda^2 \\ -(p^3-2p+1)\lambda-(p^2-1)((p-1)^2+q-1)).\end{aligned}$$

Now use Theorem 1.4 to complete the proof. □

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