Journal of Discrete Mathematics and Its Applications 8 (4) (2023) 223–237

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Research Paper

Ramanujan Cayley graphs on sporadic groups

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Academic Editor: Majid Arezoomand

Abstract. Let Γ be a $k-$regular graph with the second maximum eigenvalue λ. Then Γ is said o be Ramanujan graph if $\lambda \leq 2\sqrt{k-1}$. Let G be a finite group and $\Gamma = Cay(G,S)$ be a Cayley graph related to G. The aim of this paper is to investigate the Ramanujan Cayley graphs of sporadic groups.

Keywords: sporadic group, character table, Cayley graph, eigenvalue
Mathematics Subject Classification (2010): 05C09.

1 Introduction

Recently the theory of Ramanujan graphs has received more attention in the literate. It is a well-known fact that these graphs resolve an extremal problem in communication network theory. On the other hand, they fuse diverse branches of pure mathematics, namely, number theory, representaion theory and algebraic geometry. The aim of the present paper is to determine the Ramanujan Cayley graph in terms of a normal symmetric generating subset (or *NSGS* for briefly) where G is a sporadic group. It should be noted that computing the spectrum of Cayley graphs was started by a paper of Babai [3] in 1979 and recently, this exciting research topic is received increasing attention by mathematician, see for example [1,3,5,8,9,11,14]. Most of results of this paper are based on Theorem 2.2. In the next section, we give the necessary definitions and some preliminary results and section three contains the main results, namely, computing the Ramanujan Cayley graph of linear and sporadic groups.

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Received 1 November 2023; Revised 10 November 2023; Accepted 15 November 2023
First Publish Date: 1 December 2023

All graphs and groups considered in this paper are finite. Also all graphs are connected graphs without loops and parallel edges.

2 Definitions and Preliminaries

Let Γ be a $k-$regular graph with the second maximum eigenvalue λ. Then Γ is a Ramanujan graph if

$$\lambda \leq 2\sqrt{k-1}.$$

A symmetric subset of group G is a subset $S \subseteq G$, where $1 \notin S$ and $S = S^{-1}$. The Cayley graph $\Gamma = Cay(G,S)$ with respect to S is a graph whose vertex set is $V(\Gamma) = G$ and two vertices $x, y \in V(\Gamma)$ are adjacent if and only if $y = xs$ for an element $s \in S$. It is well-known fact that $Cay(G,S)$ is connected if and only if S generates the group G, see [4,17].

A general linear group $GL(V)$ of vector space V is the set of all $A \in End(V)$ where A is invertible. A representation of group G is a homomorphism $\alpha : G \to GL(V)$ and the degree of α is equal to the dimension of V. A trivial representation is a homomorphism $\alpha : G \to \mathbb{C}^*$ where $\alpha(g) = 1$ for all $g \in G$. Let $\varphi : G \to GL(V)$ be a representation with $\varphi(g) = \varphi_g$, the character $\chi_\varphi : G \to \mathbb{C}$ of φ is defined as $\chi_\varphi(g) = tr(\varphi_g)$. An irreducible character is the character of an irreducible representation and the character χ is linear, if $\chi(1) = 1$. We denote the set of all irreducible characters of G by $Irr(G)$. The number of irreducible characters of G is equal to the number of conjugacy classes of G and the number of linear characters of finite group G is $|G/G^{'}|$ where $G^{'}$ is the derivative subgroup of G.

A character table is a matrix whose rows and columns are correspond to the irreducible characters and the conjugacy classes of G, respectively. The study of spectrum of Cayley graphs is closely related to irreducible characters of G. If G is abelian, then the spectrum of $\Gamma = Cay(G,S)$ can easily be determined as follows.

Theorem 2.1. *Let S be a symmetric subset of abelian group G where $1 \notin S$. Then the eigenvalues of the adjacency matrix of $Cay(G,S)$ are given by*

$$\lambda_\varphi = \sum_{s\in S} \varphi(s),$$

where $\varphi \in Irr(G)$.

Let G be a finite group with symmetric subset S. We recall that S is a normal subset if and only if $S^g = g^{-1}Sg = S$, for all $g \in G$. The following theorem is implicitly contained in [7,13].

Theorem 2.2. *[7] Let α is the characteristic function on S and $\Gamma = Cay(G,S)$ be a Cayley graph on G. Let φ_k $(k = 1, \ldots, n)$ be an irreducible inequivalent representation of G. Let d_k be the degree of φ_k and ε_k denote to the eigenvalue of Γ corresponded to the linear map $\sum_{g\in G} \alpha(g)\varphi(g)$. Then*

- **i)** *the set of eigenvalues of A (adjacency matrix of $Cay(G,S)$) equal $\cup_{k=1}^{n}\{\varepsilon_k\}$; and*
- **ii)** *if the eigenvalue λ occurs with multiplicity $m_k(\lambda)$ in $\sum_{g\in G} \alpha(g)\varphi(g)$, then the multiplicity of λ in A is $\sum_{k=1}^{n} d_k m_k(\lambda)$.*

If α is a class function, then

$$\lambda_k = \frac{|G|}{d_k}\langle \alpha, \bar{\chi}_k \rangle.$$

Corollary 2.3. *Let G be a finite group with an NSGS S. Let A be the adjacency matrix of graph $\Gamma = Cay(G,S)$. Then the eigenvalues of A are given by*

$$[\lambda_\chi]^{\chi(1)^2},\ \chi \in Irr(G),$$

where $\lambda_\chi = \frac{1}{\chi(1)}\sum_{s\in S}\chi(s)$.

Thus, in a Ramanujan Cayley graph, we have

$$\sum_{s\in S}\chi(s) \le 2\chi(1)\sqrt{|S|-1}.$$

In what follows assume that

$$\delta_A(B) = \begin{cases} 1 & A \subseteq B \\ 0 & A \not\subseteq B \end{cases}.$$

Example 2.4. *[8,14] Consider the cyclic group $\mathbb{Z}_n$ in two separately cases:*
Case 1. *n is odd, thus $C_i = \{x^i, x^{-i}\}$ $(1 \le i \le \frac{n-1}{2})$ are normal symmetric subsets of $\mathbb{Z}_n$ and so*

$$S \subseteq \bigcup_{i=1}^{\frac{n-1}{2}} C_i.$$

For $0 \le j \le n-1$, $\chi_j(x^i) = \omega^{ij}$ are all irreducible characters of $\mathbb{Z}_n$, where x is a generator of $\mathbb{Z}_n$ and $\omega = e^{\frac{2\pi}{n}i}$. Hence

$$\lambda_{\chi_j} = \sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S)(\omega^{ij} + \omega^{-ij}).$$

Case 2. *n is even, hence all normal symmetric subsets are*

$$C_i = \{x^i, x^{-i}\}\ (1 \le i \le \frac{n}{2} - 2) \text{ and } C_{\frac{n}{2}-1} = \{x^{n/2}\}.$$

Therefore

$$S \subseteq \bigcup_{i=1}^{\frac{n}{2}-2} C_i.$$

Similar to the last case, we have

$$\lambda_{\chi_j} = \sum_{i=1}^{\frac{n}{2}-2} \delta_{C_i}(S)(\omega^{ij} + \omega^{-ij}) + (-1)^j \delta_{C_{\frac{n}{2}-1}}(S).$$

Example 2.5. *Consider now the dihedral group D_{2n} with the following presentation:*

$$D_{2n} = \langle a, b, a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

Here, by using Theorem 2.2, we determine the spectrum of $Cay(D_{2n}, S)$, where S is an NSGS. Let us to show the conjugacy class of $g \in G$ by g^G. In finding the number of conjugacy classes of dihedral group D_{2n}, it is convenient to consider two separated cases:

Case 1. *n is odd, then D_{2n} has precisely $\frac{1}{2}(n+3)$ conjugacy classes:*

$$\{1\}, \{a^i, a^{-i}\} \ (1 \leq i \leq (n-1)/2), \{b, ba, \cdots, ba^{n-1}\}.$$

Hence, the normal symmetric subsets of D_{2n} are

$$C_i = \{a^i, a^{-i}\}, \ (1 \leq i \leq \frac{n-1}{2}) \text{ and } C_{\frac{n+1}{2}} = b^{D_{2n}}.$$

This implies that $S \subseteq \bigcup_{i=1}^{\frac{n+1}{2}} C_i$ and so by using Table 1, we have

$$\lambda_{\chi_1} = n\delta_{C_{\frac{n+1}{2}}}(S) + 2\sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S),$$
$$\lambda_{\chi_2} = -n\delta_{C_{\frac{n+1}{2}}}(S) + 2\sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S),$$
$$\lambda_{\psi_j} = \sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S)(\varepsilon^{ij} + \varepsilon^{-ij}) \ (1 \leq j \leq \frac{n-1}{2}),$$

where $\varepsilon = e^{\frac{2\pi}{n}i}$.

Case 2. *n is even, then D_{2n} has precisely $\frac{n}{2}+3$ conjugacy classes:*

$$\{1\}, \{a^{\frac{n}{2}}\}, \{a^i, a^{-i}\}, \{ba^{2j}\}, \{ba^{2j+1}\}.$$

So, the normal symmetric subsets of D_{2n} are:

$$C_i = \{a^i, a^{-i}\}, \ (1 \leq i \leq \frac{n}{2} - 1), C_{\frac{n}{2}} = \{a^{n/2}\}, C_{\frac{n}{2}+1} = b^{D_{2n}} \text{ and } C_{\frac{n}{2}+2} = ba^{D_{2n}}.$$

Hence, $S \subseteq \bigcup_{i=1}^{\frac{n}{2}+2} C_i$ *and by using Table 2, we have*

$$\lambda_{\chi_1} = \delta_{C_{\frac{n}{2}}}(S) + \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) + \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1} \delta_{C_i}(S),$$

$$\lambda_{\chi_2} = \delta_{C_{\frac{n}{2}}}(S) - \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) + \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1} \delta_{C_i}(S),$$

$$\lambda_{\chi_3} = (-1)^{\frac{n}{2}}\delta_{C_{\frac{n}{2}}}(S) + \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) - \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1} \delta_{C_i}(S)(-1)^j,$$

$$\lambda_{\chi_4} = (-1)^{\frac{n}{2}}\delta_{C_{\frac{n}{2}}}(S) - \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) - \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1} \delta_{C_i}(S)(-1)^j,$$

$$\lambda_{\psi_j} = (-1)^j\delta_{C_{\frac{n}{2}}}(S) + \sum_{i=1}^{\frac{n}{2}-1} \delta_{C_i}(S)(\varepsilon^{ij} + \varepsilon^{-ij})\ (1 \le j \le \frac{n}{2} - 1).$$

As a special case, the minimal SNGS of group D_{2n} *is*

$$\Delta = \begin{cases} b^{D_{2n}} \cup \{a, a^{-1}\} & , \ 2|n \\ b^{D_{2n}} & , \ 2 \not| n \end{cases}.$$

Hence, the spectrum of Cayley graph $\Gamma = Cay(D_{2n}, \Delta)$ *is*

- *n is odd:*

$$\{[-n]^1, [n]^1, [0]^{2n-2}\}.$$

Since $0 \le 2\sqrt{n-1}$*, in this case* $Cay(D_{2n}, S)$ *is Ramanujan.*

- *n is even:*

$$\{[\pm n/2 \pm 2]^1, [0]^{2n-4}\}.$$

Since for $n \ge 6$*,* $\frac{n}{2} - 2 \ge 2\sqrt{\frac{n}{2}+1}$*,* $Cay(D_{2n}, S)$ *is not Ramanujan.*

g	1	a^r	b
χ_1	1	1	1
χ_2	1	1	-1
ψ_j	2	$\varepsilon^{jr} + \varepsilon^{-jr}$	0

Table 1. *The character table of group* D_{2n} *where n is odd and* $1 \le r, j \le \frac{n-1}{2}$*.*

g	1	$a^{\frac{n}{2}}$	a^r	b	ba
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^{\frac{n}{2}}$	$(-1)^r$	1	-1
χ_4	1	$(-1)^{\frac{n}{2}}$	$(-1)^r$	-1	1
ψ_j	2	$2(-1)^j$	$\varepsilon^{jr} + \varepsilon^{-jr}$	0	0

Table 2. *The character table of group D_{2n} where n is even and $1 \leq r,j \leq \frac{n}{2} - 1$.*

Since all eigenvalues of $\Gamma = Cay(D_{2n}, S)$ are symmetric with respect to the origin, according to [6, Theorem 3.2.3] Γ is bipartite.

3 Main Results

By investigating Cayley graphs, even more detailed information can be obtained. For example, the automorphism graph of a Cayley graph whose all eigenvalues are simple is an elementary $2-$group. The aim of this section is to investigate Ramanujan Cayley graph $Cay(G,S)$ via the character table of G where S is an $NSGS$ of sporadic group G.

Example 3.1. *Consider the group T_{4n} with the following presentation:*

$$T_{4n} = \langle a,b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$

The conjugacy classes of T_{4n} are

$$\{1\}, \{a^n\}, \{a^r, a^{-r}\ , 1 \leqslant r \leqslant n-1\},$$
$$\{ba^{2j}, 0 \leqslant j \leqslant n-1\}, \{ba^{2j+1}, 0 \leqslant j \leqslant n-1\}.$$

Let $S = \{a, a^{-1}, b, b^{-1}\}$.

Case 1. *n is even, then all irreducible representations of T_{4n} are as follows:*

$$id : (a,b) \to (1,1) \quad , \quad \varphi_1 : (a,b) \to (1,-1),$$
$$\varphi_2 : (a,b) \to (-1,1) \quad , \quad \varphi_3 : (a,b) \to (-1,-1)$$

and

$$\psi_k : (a,b) \to (\begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \varepsilon^{kn} & 0 \end{pmatrix})$$

where $\varepsilon = e^{\frac{2\pi i}{2n}}$ $(0 \leqslant k \leqslant n-1)$. If $\varphi_1(a, a^{-1}, b, b^{-1}) = (1,1,-1,-1)$, then we conclude thate $\lambda_1 = 0$ and if $\varphi_2(a, a^{-1}, b, b^{-1}) = (-1,-1,1,1)$, then $\lambda_2 = 0$. By regarding φ_3 we achieve $\lambda_3 = -4$. Therefor, the second maximum eigenvalue λ can be obtained from a non-linear irreducible representation. In other words

$$\lambda_k = 2\cos\frac{2k\pi}{2n} \pm (1 + \cos k\pi).$$

Case 2. *n is odd, then all irreducible characters are*

$$id : (a,b) \to (1,1), \quad \varphi_1 : (a,b) \to (-1,i),$$
$$\varphi_2 : (a,b) \to (1,-1), \quad \varphi_3 : (a,b) \to (-1,i)$$

and

$$\psi_k : (a,b) \to (\begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \varepsilon^{kn} & 0 \end{pmatrix})$$

where $\varepsilon = e^{\frac{2\pi i}{2n}} (0 \leqslant k \leqslant n-1)$. *For* $n \geqslant 6$,

$$\frac{\pi}{n} \leq \frac{\pi}{6} \Rightarrow 2\cos\frac{\pi}{n} + 2 \geq 2\cos\frac{\pi}{6} + 2 > 2\sqrt{3}.$$

This means that $Cay(G,S)$ *is not Ramanujan. Hence, in this case* $Cay(T_{4n},S)$ *is Ramanujan if and only if* $n = 1,3,5$. *Similar to the Case 1, the* $Cay(G,S)$ *is Ramanujan if and only if* $n = 1,3$. *Hence, we can verify that* $Cay(T_{4n},S)$ *is Ramanujan if and only if* $n = 1,2,3,4$.

Example 3.2. *Consider now the group* U_{6n} *with the following presentation:*

$$U_{6n} = \langle a,b | a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$$

and set $S = \{a,a^{-1},b,b^{-1}\}$, *clearly, S is not normal. For* $0 \leq j \leq n-1$ *the conjugacy classes of* U_{6n} *are as follows:*

$$\{a^{2j}\}, \{a^{2j}b, a^{2j}b^2\}, \{a^{2j+1}, a^{2j+1}b, a^{2j+1}b^2\}.$$

All irreducible representations are

$$\psi : (a,b) \to (0,-1),$$
$$\varphi_k : (a,b) \to (\varepsilon^{2k},1), 0 \leq k \leq 2n-1,$$

and

$$\psi_k : (a,b) \to (\begin{pmatrix} 0 & \varepsilon^k \\ \varepsilon^{-k} & 0 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix})$$

where $\varepsilon = e^{\frac{2\pi i}{2n}}, \omega = e^{\frac{2\pi i}{3}}$. *Hence we have*

$$\lambda_k = \psi_k(a) + \psi_k(a^{-1}) + \psi_k(b) + \psi_k(b^{-1}) = \varepsilon^{2k} + \varepsilon^{-2k} + 2 = 2 + 2\cos\frac{2k\pi}{2n}$$

and for non-linear representation we also have

$$\sum_{g \in S} \psi_k = \begin{pmatrix} \omega + \omega^2 & \varepsilon^k + \varepsilon^{-k} \\ \varepsilon^k + \varepsilon^{-k} & \omega + \omega^2 \end{pmatrix} = \begin{pmatrix} -1 & 2\cos\frac{k\pi}{n} \\ 2\cos\frac{k\pi}{n} & -1 \end{pmatrix}.$$

Thus

$$\mu_k = -1 \pm 2\cos\frac{k\pi}{n}.$$

One can see that $|\mu_k| < 2\sqrt{3}$ *and for* $n \geq 9$ *and* $k = 1$, *we have*

$$2 + 2\cos\frac{2\pi}{n} \geq 2 + 2\cos\frac{2\pi}{9} > 2\sqrt{3}.$$

On the other hand, for $n \leq 8$, $\lambda < 2\sqrt{3}$ *and thus* $Cay(U_{6n},S)$ *is Ramanujan if and only if* $n \leq 8$.

Example 3.3. *Suppose the group V_{8n} has the following presentation:*

$$V_{8n} = \langle a,b | a^{2n} = b^4 = 1, aba = b^{-1}, ab^{-1}a = b \rangle.$$

For $1 \le r \le \frac{n-1}{2}$ and $0 \le s \le n-1$, the conjugacy classes of V_{8n} are as follows:

$$\{1\}\{b^2\}, \{a^{2r}, a^{-2r}\}, \{a^{2r}b^2, a^{-2r}b^2\}, \{a^{2s+1}, a^{-2s-1}b^2\},$$
$$\{a^{2l}b, a^{2l}b^3 | 0 \le l \le n-1\}, \{a^{2l+1}b, a^{2l+1}b^3 | 0 \le l \le n-1\}.$$

It is clear that $S = \{a, a^{-1}, b, b^{-1}\}$ is not normal and all irreducible representations of V_{8n} are as follows:

$$f_1 : (a,b) \to (1,1)\ ,\ f_2 : (a,b) \to (-1,1)\ ,\ f_3 : (a,b) \to (-1,-1),$$
$$\psi_k : (a,b) \to (\begin{pmatrix} \varepsilon^{2k} & 0 \\ 0 & -\varepsilon^{-2k} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}), 0 \le k \le n-1, \varepsilon = e^{\frac{2\pi i}{2n}},$$
$$\varphi_k : (a,b) \to (\begin{pmatrix} \varepsilon^{k} & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}), 1 \le k \le \frac{n-1}{2}.$$

Hence,

$$\sum_{g \in S} \psi_k = \begin{pmatrix} \varepsilon^{2k} + \varepsilon^{-2k} & 0 \\ 0 & -(\varepsilon^{2k} + \varepsilon^{-2k}) \end{pmatrix}.$$

This yields that $\lambda_k = \pm 2\cos\frac{2k\pi}{n}$ and so $|\lambda_k| =< 2\sqrt{3}$. On the other hand,

$$\sum_{g \in S} \psi_k = \begin{pmatrix} \varepsilon^{k} + \varepsilon^{-k} & 0 \\ 0 & \varepsilon^{k} + \varepsilon^{-k} \end{pmatrix}$$

implies that $\lambda_k = 2\cos\frac{k\pi}{n}$ and thus $|\lambda_k| =< 2\sqrt{3}$. Therefor, $Cay(V_{8n}, S)$ is Ramanujan.

3.1 Linear Groups

Let $V(n,\mathbb{F})$ denotes the $n-$dimensional vector space over a field $\mathbb{F}$. A transvection is a linear transformation T on $V(n,\mathbb{F})$ with eigenvalues equal to 1 and satisfying $rank(T - I_n) = 1$, where I_n is the edentity transformation on $V(n,\mathbb{F})$. In matrix language a transvection $A_{ij}(\alpha)$ where $i \neq j$ and $\alpha \in \mathbb{F}$, is a matrix different from the identity that it has α in the $(i,j)-$th position. It turns out that all transvections are elements of $SL(n,\mathbb{F})$.

Proposition 3.4. *[2] For integers i , j, the set $\mathcal{A}_{ij} = \{A_{ij}(\alpha) \mid \alpha \in \mathbb{F}\}$ forms a subgroup of $SL(n,\mathbb{F})$.*

The subgroups defined in this way are refer as the root subgroup of $GL(n,\mathbb{F})$. By Proposition 3.4, the group $SL(n,\mathbb{F})$ is generated by the root subgroups $\mathcal{A}_{ij}$. In other words,

$$SL(n,\mathbb{F}) = \langle \mathcal{A}_{ij}\ :\ 1 \leqslant i \neq j \leqslant n \rangle.$$

By using Proposition 3.4 the group $GL(n,\mathbb{F})$ is also generated by the set of all invertible diagonal matrices and all transvections.

Theorem 3.5. *All transvections are conjugate in $GL(n,q)$ and if $n \geqslant 3$, then all transvections are conjugate in $SL(n,q)$.*

Conjugacy classes of $SL(2,q)$ (q is odd)

The number of classes of $SL(2,q)$ is $q+4$ (see [2]) and two following cases hold:
Case 1. q is odd, the character table and conjugacy classes of $SL(2,q)$ is as reported in Table 3 and Table 4.

Type	Rep g	No. CC	$\|[g]\|$
$\mathcal{T}_0^{(1)}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	1	1
$-\mathcal{T}_0^{(1)}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	1	1
$\mathcal{T}_{01}^{(2)}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	1	$\frac{q^2-1}{2}$
$-\mathcal{T}_{01}^{(2)}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	1	$\frac{q^2-1}{2}$
$\mathcal{T}_{0\varepsilon}^{(2)}$	$\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$	1	$\frac{q^2-1}{2}$
$-\mathcal{T}_{0\varepsilon}^{(2)}$	$\begin{pmatrix}-1&-\varepsilon\\0&1\end{pmatrix}$	1	$\frac{q^2-1}{2}$
$\mathcal{T}_{k,-k}^{(3)}$	$\begin{pmatrix}\alpha&0\\0&\alpha^{-1}\end{pmatrix}$	$\frac{q-3}{2}$	$q(q+1)$
$-\mathcal{T}_{k}^{(4)}$	$\begin{pmatrix}0&1\\-1&-(r+r^q)\end{pmatrix}$	$\frac{q-1}{2}$	$q(q-1)$

Table 3. The conjugacy classes of $SL(2,q)$, q is odd:

In table 3, by No. CC we mean the number of conjugacy classes of prescribed type of classes and by Rep g we mean the representation of g.

Class	$\mathcal{T}_0^{(1)}$	$-\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$-\mathcal{T}_{01}^{(2)}$
Rep$g$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&-1\\0&-1\end{pmatrix}$
$\|[g]\|$	1	1	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$
λ	1	1	1	1
ψ	q	q	0	0
$\psi_{k,1}$	$q+1$	$(-1)^{k+1}(q+1)$	1	1
π_k	$q-1$	$(-1)^k(q-1)$	-1	$(-1)^{k+1}$
ξ_1	$\frac{q+1}{2}$	$\theta\frac{(q+1)}{2}$	$\frac{1}{2}(1+\sqrt{\theta q})$	$\frac{\theta}{2}(1+\sqrt{\theta q})$
ξ_2	$\frac{q+1}{2}$	$\theta\frac{(q+1)}{2}$	$\frac{1}{2}(1-\sqrt{\theta q})$	$\frac{\theta}{2}(1-\sqrt{\theta q})$
υ_1	$\frac{q-1}{2}$	$-\theta\frac{(q-1)}{2}$	$\frac{1}{2}(-1+\sqrt{\theta q})$	$\frac{-\theta}{2}(1+\sqrt{\theta q})$
υ_2	$\frac{q-1}{2}$	$-\theta\frac{(q-1)}{2}$	$\frac{1}{2}(-1-\sqrt{\theta q})$	$\frac{-\theta}{2}(-1-\sqrt{\theta q})$

continued:

Class	$\mathcal{T}_{0\varepsilon}^{(2)}$	$-\mathcal{T}_{0\varepsilon}^{(2)}$	$\mathcal{T}_{k,-k}^{(3)}$	$-\mathcal{T}_{k}^{(4)}$
Rep$g$	$\begin{pmatrix}1 & \varepsilon\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & -\varepsilon\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -1 & -(r+r^q)\end{pmatrix}$
$\lvert[g]\rvert$	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$	$q(q+1)$	$q(q-1)$
λ	1	1	1	1
ψ	0	0	1	-1
$\psi_{k,1}$	1	$(-1)^{(k+1)}$	$\varepsilon^{(k-1)}+\varepsilon^{-(k-1)}$	0
π_k	-1	$(-1)^{k+1}$	0	$-(r^k+r^{kq})$
ξ_1	$\frac{1}{2}(1-\sqrt{\theta q})$	$\frac{\theta}{2}(1-\sqrt{\theta q})$	$(-1)^k$	0
ξ_2	$\frac{1}{2}(1+\sqrt{\theta q})$	$\frac{\theta}{2}(1+\sqrt{\theta q})$	$(-1)^k$	0
v_1	$\frac{1}{2}(-1-\sqrt{\theta q})$	$\frac{-\theta}{2}(-1-\sqrt{\theta q})$	0	$(-1)^{m+1}$
v_2	$\frac{1}{2}(-1+\sqrt{\theta q})$	$\frac{-\theta}{2}(-1+\sqrt{\theta q})$	0	$(-1)^{m+1}$

Table 4. The character table of $SL(2,q)$, q is odd:

Let $A=\begin{pmatrix}1 & \varepsilon^{2t+1}\\ 0 & 1\end{pmatrix}, t\neq 0$, then for $B=\begin{pmatrix}\varepsilon^t & 0\\ 0 & \varepsilon^{-t}\end{pmatrix}$ we have $B^{-1}AB=\begin{pmatrix}1 & \varepsilon\\ 0 & 1\end{pmatrix}$ and $A\in\mathcal{T}_{0\varepsilon}^{(2)}$. Similarly for $\begin{pmatrix}1 & \varepsilon^{2t}\\ 0 & 1\end{pmatrix}$, we have

$$\begin{pmatrix}\varepsilon^t & 0\\ 0 & \varepsilon^{-t}\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\varepsilon^t & 0\\ 0 & \varepsilon^{-t}\end{pmatrix}^{-1}=\begin{pmatrix}1 & \varepsilon^{2t}\\ 0 & 1\end{pmatrix}.$$

Also all matrixes in the form $\begin{pmatrix}1 & 0\\ \varepsilon^{2t+1} & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ \varepsilon^{2t} & 1\end{pmatrix}$ belong to $\mathcal{T}_{01}^{(2)}$ and $\mathcal{T}_{0\varepsilon}^{(2)}$, since

$$\begin{pmatrix}0 & -\varepsilon^k\\ \varepsilon^{-k} & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ \varepsilon^{2t+1} & 1\end{pmatrix}\begin{pmatrix}0 & \varepsilon^k\\ -\varepsilon^{-k} & 0\end{pmatrix}=\begin{pmatrix}1 & \varepsilon\\ 0 & 1\end{pmatrix}\ ;\ k=(\frac{q-1}{4})-t$$
$$\begin{pmatrix}0 & -\varepsilon^k\\ \varepsilon^{-k} & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ \varepsilon^{2t} & 1\end{pmatrix}\begin{pmatrix}0 & \varepsilon^k\\ -\varepsilon^{-k} & 0\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\ ;\ k=(\frac{q-1}{4})-t$$

Thus $S=\mathcal{T}_{01}^{(2)}\cup\mathcal{T}_{0\varepsilon}^{(2)}$ is a genarator of $G=SL(2,q)$. The character table of G is

Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_{01}^{(2)}$	$\mathcal{T}_{0\varepsilon}^{(2)}$
Rep$g$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$
$\|[g]\|$	1	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$
λ	1	1	1
ψ	q	0	0
$\psi_{k,1}$	$q+1$	1	1
π_k	$q-1$	-1	-1
ξ_1	$\frac{q+1}{2}$	$\frac{1}{2}(1+\sqrt{\theta q})$	$\frac{1}{2}(1-\sqrt{\theta q})$
ξ_2	$\frac{q+1}{2}$	$\frac{1}{2}(1-\sqrt{\theta q})$	$\frac{1}{2}(1+\sqrt{\theta q})$
v_1	$\frac{q-1}{2}$	$\frac{1}{2}(-1+\sqrt{\theta q})$	$\frac{1}{2}(-1-\sqrt{\theta q})$
v_2	$\frac{q-1}{2}$	$\frac{1}{2}(-1-\sqrt{\theta q})$	$\frac{1}{2}(-1+\sqrt{\theta q})$

where for $q=4n+1$ we have $\theta=1$ and for $q=4n+3$ we have $\theta=-1$. Therefor all eigenvalues of $Cay(G,S)$ are

$$\begin{aligned}
\mu_1 &= q^2-1=|S|,\\
\mu_2 &= 0,\\
\mu_3 &= \frac{1}{q+1}(q^2-1)=q-1,\\
\mu_4 &= \frac{-1}{q-1}(q^2-1)=-(q+1),\\
\mu_5 &= \frac{2}{q+1}\frac{q^2-1}{2}=q-1=\mu_6,\\
\mu_7 &= \frac{-2}{q-1}\frac{q^2-1}{2}=-q-1=\mu_7.
\end{aligned}$$

Hence, the spectrum of $Cay(SL(2,q),S)$ is $\{[0],[-q-1],[q+1],[q^2-1]\}$. Since, $\lambda=q+1$, we can deduce that $Cay(SL(2,q),S)$ is Ramanujan.

Case 2. The number of conjugacy classes of $SL(2,q)$ where $2|q$ is $q+1$. see [2, proposition

4.4.7]. On the other hand, the character table of $SL(2,q)$ is as reported in Table 5.

The conjugacy classes and character table of $SL(2,q)$, q is even				
Class	$\mathcal{T}_0^{(1)}$	$\mathcal{T}_0^{(2)}$	$\mathcal{T}_{k,-k}^{(3)}$	$\mathcal{T}_k^{(4)}$
Rep$g$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}\alpha&0\\0&\alpha^{-1}\end{pmatrix}$	$\begin{pmatrix}0&1\\1&r+r^q\end{pmatrix}$
No. of CC	1	1	$\frac{q-2}{2}$	$\frac{q}{2}$
$\lvert[g]\rvert$	1	q^2-1	$q(q+1)$	$q(q-1)$
λ	1	1	1	1
ψ	q	0	1	-1
$\psi_{k,0}$	$q+1$	1	$\alpha^k+\alpha^{-k}$	0
π_k	$q-1$	-1	0	$-(r^k+r^{kq})$

Table 5. The character table of $SL(2,q)$, q is even.

Let q be even. we have

$$\begin{pmatrix}\varepsilon^{\frac{q}{2}}&0\\0&\varepsilon^{-\frac{q}{2}}\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}\varepsilon^{\frac{q}{2}}&0\\0&\varepsilon^{-\frac{q}{2}}\end{pmatrix}^{-1}=\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$$

and

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}^{-1}=\begin{pmatrix}1&-\varepsilon\\0&1\end{pmatrix}=\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}.$$

It is not difficult to see that $S=\mathcal{T}_0^{(2)}$ is a ganarator of $G=SL(2,q)$ and eigenvalues of $Cay(G,S)$ are

$$\begin{aligned}
\mu_1 &= q^2-1=|S|,\\
\mu_2 &= 0,\\
\mu_3 &= \frac{1}{q+1}(q^2-1)=q-1,\\
\mu_4 &= \frac{-1}{q-1}(q^2-1)=-(q+1).
\end{aligned}$$

Therefore $\lambda=q+1$ and hence $Cay(G,S)$ is Ramanujan.

3.2 Mathieu Groups

We find from GAP, the conjugacy classes of mathieu group $G=M(9)$ are

$$A=\{()^G,(2,3,8,6)(4,7,5,9)^G,(2,4,8,5)(3,9,6,7)^G,$$
$$(2,7,8,9)(3,4,6,5)^G,(2,8)(3,6)(4,5)(7,9)^G,(1,2,8)(3,9,4)(5,7,6)^G\}.$$

Thus, the eigenvalues of $Cay(G,S)$, where $S = a^G \cup b^G$, $a^G, b^G \in A$ are

$$[36,-36,0,0,0,0], \quad [36,0,-36,0,0,0],$$
$$[27,-9,-9,27,-9,0], \quad [26,-10,-10,26,8,1],$$
$$[36,0,0,-36,0,0], \quad [27,-9,27,-9,-9,0],$$
$$[26,-10,26,-10,8,-1], \quad [27,27,-9,-9,-9,0],$$
$$[26,26,-10,-10,8,-1], \quad [17,17,17,17,-1,-1].$$

It yields that $Cay(G,S)$ is Ramanujan. In the special case

$$S = \{(2,8)(3,6)(4,5)(7,9)^G, (1,2,8)(3,9,4)(5,7,6)^G\}$$

and S is a set with minimum size.

The conjugacy classes of mathieu group $G = M(10)$ are as follow,

$$\{()^G, (3,4,9,7)(5,8,6,10)^G, (3,5,9,6)(4,10,7,8)^G,$$
$$(3,9)(4,7)(5,6)(8,10)^G, (2,3,9)(4,10,5)(6,8,7)^G,$$
$$(1,2)(3,4,5,10,9,7,6,8)^G, (1,2)(3,7,5,8,9,4,6,10)^G,$$
$$(1,2,3,7,6)(4,8,5,9,10)^G\}.$$

The eigenvalues of $Cay(G,S)$ for $S = \{a^G\}$ where $a^G \in A$ are

$$[1,1,1,1,1,1,1,1], [180,-180,-20,20,0,0,0,0],$$
$$[90,90,10,10,-18,0,0,0], [45,45,5,5,9,-9,-9,0],$$
$$[80,80,0,0,8,8,8,-10],$$
$$[90,-90,10,-10,0,-9*E(8)-9*E(8)^3, 9*E(8)+9*E(8)^3, 0],$$
$$[90,-90,10,-10,0,9*E(8)+9*E(8)^3, -9*E(8)-9*E(8)^3, 0],$$
$$[144,144,-16,-16,0,0,0,9].$$

For $S = \{(3,9)(4,7)(5,6)(8,10)^G\}$ the eigenvalues of $M(10)$ are, $\{45,45,5,5,9,-9,-9,0\}$ and in this case $Cay(G,S)$ is Ramanujan. The conjugacy classes of mathieu group $G = M(11)$ are also as follows

$$A = \{()^G, (1,11,2,5,3,8,10,9,7,6,4)^G, (1,4,6,7,9,10,8,3,5,2,11)^G,$$
$$(2,5)(3,10)(4,9)(7,8)^G, (2,7,5,8)(3,9,10,4)^G,$$
$$(1,5,6,11,7,8,2,10)(4,9)^G, (1,10,2,8,7,11,6,5)(4,9)^G,$$
$$(1,11,6)(2,4,3)(5,9,10)^G, (1,6,11)(2,10,4,5,3,9)(7,8)^G,$$
$$(1,5,8,3,10)(2,11,7,9,6)^G\}.$$

If $S = \{(1,4,6,11,8,7,10,2,3,9,5)^G, (1,3,4)(2,10)(5,7,11,6,9,8)^G, (2,6,10,5)(7,11,9,8)^G\}$, then all eigenvalues of $Cay(G,S)$ are

$$[3030,-6,60,60,-90,45*E(11)^2+45*E(11)^6+45*E(11)^7+45*E(11)^8+45*E(11)^{10},$$
$$45*E(11)+45*E(11)^3+45*E(11)^4+45*E(11)^5+45*E(11)^9, 30, 38, -42].$$

Since $2\sqrt{k-1} = 2\sqrt{3030-1} = 110$, we have

It yields that $\lambda = 90$ and so $Cay(G,S)$ is Ramanujan.

3.3 Suzuki Group

Following Suzuki [18], the group G is called a ZT-group if G acts on set Ω in such a way that, (1) G is a doubly transitive group on $1+N$ symbols. (2) The identity is the only element which leaves three distinct symbols invariant, (3) G contains no normal subgroup of order $1+N$, and (4) N is even. Suzuki [18] showed that for each prime power $q = 2^{2s+1}$, there is a unique ZT-group Sz(q) of order $q^2(q-1)(q^2+1)$ which is called later the Suzuki group. This group is simple, when $q > 2$. Suppose that a is symbol on which G acts and $H = G_a$. By [18], it follows from the conditions (1) and (2) that H is a Frobenius group on $\Omega \setminus \{a\}$. Apply a well-known result of Frobenius to deduce that H contains a regular normal subgroup Q of order N such that every non-identity element of Q leaves only the symbol a invariant. Supoose $b \in \Omega \setminus \{a\}$ and $K = H_b$. Suppos $x \in N_G(K)$ is involution. Then, it is well-known that Suzuki groups are containing two elements y and z such that y is an involution and $xyx = z^{-1}xz$, and three cyclic subgroups A_0, A_1 and A_2 of order $q-1$, $q+r-1$ and $q-r+1$, respectively. The conjugacy classes of $Sz(q)$ are

$$\{e\}, y^{Sz(q)}, z^{Sz(q)}, (z^{-1})^{Sz(q)} b_0^{Sz(q)}, b_1^{Sz(q)}, b_2^{Sz(q)}$$

which are of lengths 1, $(q-1)(q^2+1)$ $\frac{1}{2}(q-1)(q^2+1)$, $\frac{1}{2}(q-1)(q^2+1)$, $q^2(q-1)(q+r+1)$, $q^2(q+r+1)(q-r+1)$, and $q^2(q-1)(q-r+1)$, respectively. Here, b_0, b_1 and b_2 are non-identity alements of A_i, $i = 0,1,2$, respectively. Note that there are $\frac{q-r}{2}$ $\frac{q}{2}-1$ and $\frac{q+r}{4}$ conjugacy classes of types $b_0^{Sz(q)}, b_1^{Sz(q)}$ and $b_2^{Sz(q)}$, respectively. Consider the Suzuki group $Sz(q)$ whit $q = 2^{2s+1}, r = 2^{s+1}$ and $s \geq 1$. The conjugacy class $S = y^{Sz(q)}$ and the normal subset $T = z^{Sz(q)} \cup (z^{-1})^{Sz(q)}$ are minimal $NSGS$ and second minimal $NSGS$ of $Sz(q)$, respectively. Moreover, $|S| = (q-1)(q^2+1)$, $|T| = q(q-1)(q^2+1)$ and the simple eigenvalues of $Cay(Sz(q),S)$ and $Cay(Sz(q),T)$ are $|S|$ and $|T|$, respectively. The Cayley graph $Cay(Sz(q),S)$ has eigenvalues:

$$0, -(q^2+1), (q-1), \frac{(1+q^2)(r-1)}{q-r+1}, \frac{-(1+q^2)(r+1)}{q+r+1}.$$

Thus $|1+q^2| \nless 2\sqrt{|S|-1}$ and $Cay(Sz(q),S)$ is not Ramanujan graph. The Cayley graph $Cay(Sz(q),T)$ has eigenvalues:

$$0, q(q-1), \frac{-q(q^2+1)}{q-r+1}, \frac{-q(1+q^2)}{q+r+1}.$$

in this case $Cay(Sz(q), S)$ is Ramanujan.

Funding: This research received no external funding.
Conflicts of Interest: The author declares no conflicts of interest.

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Citation: S. Mehry, Ramanujan Cayley graphs on sporadic groups, J. Disc. Math. Appl. 8(4) (2023) 223–237.
https://10.22061/JDMA.2023.10294.1062