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Review Paper

A survey on symmetric division degree index

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Abstract. The symmetric division degree (SDD) index is one the 148 discrete Adriatic indices, introduced by Vukicević et al. as a remarkable predictor of total surface area of polychlorobiphenyls. The SDD index has already been proved a valuable index in the $QSPR/QSAR$ studies. This paper is essentially a survey of known results about bounds for SDD index of graphs.

Keywords: symmetric division degree index, extremal graphs, vertex-degree-based index
Mathematics Subject Classification (2010): 05C09.

1 Introduction

Let $G = (V, E)$ be a simple connected graph with $n \geq 2$ vertices and $m = |E|$ edges. The maximum vertex degree is denoted by Δ and the minimum by δ. For the edge $e = v_i v_j$, the degree of edge e is $d(e) = d_i + d_j - 2$, $\Delta_e = \max_{k=1}^{m} d(e_k) + 2$ and $\delta_e = \min_{k=1}^{m} d(e_k) + 2$. A graph is d-regular if all its vertices have degree d. The distance between two vertices u and v, denoted by $d(u, v)$, is defined to be the length of a shortest path joining u and v in G. The diameter of G, denoted by $d(G)$, is the maximum distance over all pairs of vertices in G. A graph G is said to be respectively a tree, a unicyclic graph, and a bicyclic graph if and only if $m = n - 1$, n, and $n + 1$. Let P_n and S_n be the n-vertex path and the n-vertex star, respectively. By S_n^+ we mean a unicycle graph constructed from the star graph S_n with an additional edge. A chemical (molecular) graph is a graph with maximum degree no more than four.

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Molecular descriptors, which are numerical functions of molecular structure, play an important role in mathematical chemistry. They are used in $QSAR$ and $QSPR$ studies to study and predict biological or chemical properties of molecules [5]. Topological indices, which are numerical functions of the underlying molecular graph, are an important group of these descriptors. The topological index that depends on the degrees of the vertices of the graph G, is known as the vertex-degree-based (VDB) index. Similarly, an edge-degree-based (EDB) index is introduced. The symmetric division degree index of graph G, $SDD(G)$, was introduced by Vukičević and Gašperov in [39], is a VDB index and one of the 148 so-called Adriatic indices, with a good predictive power for the total surface area of polychlorobiphenyl [39]. It is defined as

$$SDD = SDD(G) = \sum_{i\sim j} \frac{d_i^2 + d_j^2}{d_j d_i},$$

where d_i is a degree of vertex v_i.

Symmetric division degree index as an applicable and usable topological index whose quality is higher than some of the popular VDB indices, especially the geometric-arithmetic index [11]. In recent years and due to the importance of the SDD, many studies have been conducted on this index. In some works, the use of index in predicting the properties of chemical structures has been investigated [11, 13, 25, 28, 34], and some others, have found extremal graphs i.e graphs with minimum and maximum SDD index of different categories of graphs [2, 15, 20, 23, 24, 30, 36, 38, 40]. This text provides a review of previous researches and main results on bounds on the SDD index. Before proceeding, we recall the concepts of some well-known topological indices.

Two VDB topological indices, the first and the second Zagreb indices, M_1 and M_2, were defined in [17, 19] as

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2 = M_2(G) = \sum_{i\sim j} d_i d_j.$$

Since $M_1 = \sum_{i=1}^{m}(d(e_i) + 2)$, this index can also be considered as an EDB topological index [26].

The Hyper-Zagreb index, HM, was defined in [35] as

$$HM = HM(G) = \sum_{i\sim j} (d_i + d_j)^2.$$

The first and the second multiplicative Zagreb indices, Π_1 and P_2, were defined in [16] as

$$\Pi_1 = \Pi_1(G) = \prod_{i=1}^{n} d_i^2 \quad \text{and} \quad \Pi_2 = \Pi_2(G) = \prod_{i\sim j} d_i d_j.$$

Also, the multiplicative sum Zagreb index, Π_1^*, was introduced in [7] as

$$\Pi_1^* = \Pi_1^*(G) = \prod_{i\sim j} (d_i + d_j).$$

A so-called inverse sum indeg index, ISI, was defined in [38] as a significant predictor of total surface area of octane isomers. The inverse sum indeg index is defined as

$$ISI = ISI(G) = \sum_{i\sim j} \frac{d_i d_j}{d_i + d_j}.$$

The geometric-arithmetic index, GA, and arithmetic-geometric index AG, were introduced in [30,24], are defined as

$$GA = GA(G) = \sum_{i\sim j} \frac{2\sqrt{d_i d_j}}{d_i + d_j},$$

$$AG = AG(G) = \sum_{i\sim j} \frac{d_i + d_j}{2\sqrt{d_i d_j}}.$$

The general Randić index, R_α, was defined in [33] as

$$R_\alpha = R_\alpha(G) = \sum_{i\sim j} (d_i d_j)^\alpha.$$

Another VDB index that was mentioned and introduced in [1,18], is the sigma index, σ. It is defined as

$$\sigma = \sigma(G) = \sum_{i\sim j} (d_i - d_j)^2.$$

The so-called forgotten index was introduced in [12], is defined as

$$F = F(G) = \sum_{i=1}^{n} d_i^3 = \sum_{i\sim j} d_i^2 + d_j^2.$$

Ernesto Estrada et al. [9] proposed a new topological index, named atom-bond connectivity (ABC) index. It displays an excellent correlation with the heat of formation of alkanes [8,9]. This index is defined as follows

$$ABC = ABC(G) = \sum_{i\sim j} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}.$$

The inverse degree index ID was introduced in [10] as

$$ID = ID(G) = \sum_{i=1}^{n} \frac{1}{d_i}.$$

Klein and Randić [21] defined the Kirchhoff index of graph G, Kf, as

$$Kf = Kf(G) = \sum_{i<j} r_{ij},$$

where r_{ij} is the resistance distance between the vertices i and j of a simple connected graph G.

The inverse symmetric division deg $(ISDD)$ index, was proposed by Ghorbani et al. [14] which is given by the formula

$$ISDD = ISDD(G) = \sum_{i\sim j} \frac{d_i d_j}{d_j^2 + d_i^2}.$$

In this text, we have listed together almost all the obtained bounds for SDD. In the first two sections, we have presented the general lower and upper bounds, respectively. In the third section, we have given the bounds for special graphs such as trees and molecular graphs. And in Section 4, we have compared the existing bounds and introduced the best upper and lower bounds.

2 Lower bounds for SDD

In this section, we present the lower bounds for the SDD index in terms of some other topological indices and graph parameters.

In [4], Das et al. gave the following six theorems, contain lower bounds for the SDD index of graphs in the term of $M_1(G)$, $M_2(G)$, $ISI(G)$, $GA(G)$ and $R_{-1}(G)$.

Theorem 2.1. *[4] Let G be a connected graph of order n with $m \geq 1$ edges. Then*

$$SDD \geq \frac{M_1^2}{M_2} - 2m$$

with equality holding if and only if G is a regular graph or a semi-regular bipartite graph.

The following bounds are also given in [4], which are the result of Theorem 2.1, although they are weaker than it.

$$SDD \geq \frac{m\delta_e M_1}{M_2} - 2m \geq \frac{m^2\delta_e^2}{M_2} - 2m,$$

$$SDD \geq \frac{2m^2 M_1}{M_2 H} - 2m \geq \frac{4m^4}{M_2 H^2} - 2m,$$

$$SDD \geq \frac{M_1^2}{\Delta_e ISI} - 2m.$$

Theorem 2.2. *[4] Let G be a connected graph of order n with $m \geq 2$ edges. Then*

$$SDD \geq \frac{m^2 \Delta_e \delta_e}{(\Delta_e + \delta_e) ISI - M_2} - 2m$$

with equality holding if and only if $\frac{d_i}{d_j} + \frac{d_j}{d_i} = \frac{d_k}{d_l} + \frac{d_l}{d_k}$ for any edges $v_i v_j, v_k v_l \in E(G)$ and $d_i + d_j = \Delta_e$ or δ_e, for any edge $v_i v_j \in E(G)$.

Theorem 2.3. *[4] Let G be a connected graph with $m \geq 2$ edges. Then*

$$SDD \geq \frac{m^2 \delta_e}{ISI} - 2m.$$

Equality holds if and only if G is a regular graph or a semi-regular bipartite graph.

Theorem 2.4. *[4] Let G be a connected graph of order n with $m \geq 1$ edges. Then*

$$SDD \geq \frac{4m^3}{GA^2} - 2m \tag{1}$$

with equality holding if and only if $\frac{d_i}{d_j} + \frac{d_j}{d_i} = \frac{d_k}{d_l} + \frac{d_l}{d_k}$ for any edges $v_i v_j, v_k v_l \in E(G)$.

Theorem 2.5. *[4] Let G be a connected graph of order n with $m \geq 2$ edges. Then*

$$SDD \geq \frac{4m^4}{(m-1)GA^2} - \frac{m\left(\Pi_1^*\right)^{\frac{2}{m}}}{(m-1)(\Pi_2)^{\frac{1}{m}}} - 2m \tag{2}$$

with equality holding if and only if $\frac{d_i}{d_j} + \frac{d_j}{d_i} = \frac{d_k}{d_l} + \frac{d_l}{d_k}$ for any edges $v_i v_j, v_k v_l \in E(G)$.

Theorem 2.6. *[4] Let G be a connected graph of order n with $m \geq 2$ edges. Then*

$$SDD \geq \frac{m}{m-1}\left(2(m+1) - \frac{(\Pi_1^*)^{\frac{2}{m}}}{(\Pi_2)^{\frac{1}{m}}}\right).$$

Equality holds if and only if G is a regular graph.

Theorem 2.7. *[4] Let G be a connected graph of order n with $m \geq 1$ edges. Then*

$$SDD \geq \frac{n^2}{R_{-1}} - 2m. \tag{3}$$

Equality holds if and only if G is a regular graph or a bipartite semi-regular graph.

Furtula et al. in [11], obtained the following five lower bounds in terms of indices M_1, σ, GA and ABC.

Theorem 2.8. *[11] Let G be a graph of order n with m edges and maximum degree Δ. Then*

$$SDD \geq m\left[\left(\frac{M_1}{m\Delta}\right)^2 - 2\right],$$

with equality holding if and only if G is a regular graph.

Theorem 2.9. *[11] Let G be a graph of order n with m edges and maximum degree Δ. Then*

$$SDD \geq \frac{\sigma}{\Delta^2} + 2m,$$

with equality holding if and only if G is a regular graph.

Theorem 2.10. *[11] Let G be a connected graph of order n. Then*

$$SDD \geq 2GA,$$

the equality holds if and only if G is a regular graph.

Theorem 2.11. *[11] Let G be a graph of order n with m edges. Then*

$$SDD \geq \frac{2m^2}{GA}$$

with equality holding if and only if each connected component of G is a regular graph.

Theorem 2.12. *[11] Let G be a connected graph of order $n > 2$. Then*

$$SDD \geq \frac{3}{2}ABC.$$

A new lower bounds for SDD has been found by Ghorbani et al. in [14].

Theorem 2.13. *[14] Let G be a graph of order n with m edges. Then*

$$SDD \geq \frac{4AG^2}{m} - 2m. \tag{4}$$

Theorem 2.14. *[14] Let G be a graph of order n with m edges. Then*

$$SDD \geq \frac{m^2}{ISDD}. \tag{5}$$

In addition, if either G is regular or edge-transitive, the equality holds.

Theorem 2.15. *[14] Let G be a graph with m edges. Then*

$$SDD \geq ISDD + \frac{3m}{2},$$

with equality if and only if G is regular.

Theorem 2.16. *[15] Let G be a simple connected graph with order n, size m, p pendent vertices, maximum degree Δ and minimum non-pendent vertex degree δ_1. Then*

$$SDD(G) \geq p\left(\frac{\delta_1^2 + 1}{\delta_1}\right) + 2(m - p).$$

For the regular and star graph equality hold.

Theorem 2.17. *[15] Let G be a simple connected graph with order n, size m, maximum degree Δ and minimum degree δ. Then*

$$SDD(G) \geq \sqrt{\frac{n^2 HM}{m} - 4m^2\left(\frac{\Delta}{\delta} - \frac{\delta}{\Delta}\right)^2} - 2m.$$

Recently, Liu and Huang [22] have shown that the only graph with $0 < SDD(G) \leq 4$ is K_2 with $SDD(K_2) = 2$. If $4 < SDD(G) \leq 6$, then $G \in \{S_3, C_3\}$ with $SDD(S_3) = 5$ and $SDD(C_3) = 6$, and if $6 < SDD(G) \leq 8$, then G is P_4 or C_4 with $SDD(P_4) = 7$ and $SDD(C_4) = 8$. Also, they have obtained the new bounds for the SDD in the terms of VDB indices the inverse degree index, ID, the first Zagreb index, M_1, the second Zagreb index, M_2 and the forgotten index F.

Theorem 2.18. *[22] Let G be a graph of order n with minimum degree δ. Then*

$$SDD \geq \delta^2 ID,$$

with equality if and only if G is regular.

Theorem 2.19. *[22] Let G be a graph of order n with m edges. Then*

$$SDD \geq \frac{2m^2}{M_2} \quad and \quad SDD \geq \frac{4m^2}{F},$$

with both equalities if and only if G is regular.

Theorem 2.20. *[22] Let G be a graph of order n with m edges, maximum degree Δ and minimum degree δ. Then for $\alpha > 0$*

$$SDD \geq \frac{2\delta^2 m^{\frac{\alpha+1}{\alpha}}}{(M_2^{\alpha})^{\frac{1}{\alpha}}} \quad and \quad SDD \geq \frac{\delta^2 (2m)^{\frac{\alpha+1}{\alpha}}}{\Delta (M_1^{\alpha})^{\frac{1}{\alpha}}}.$$

with both equalities if and only if G is regular.

The following three bounds has been found by Vasilyev in [37].

Theorem 2.21. *[37] Let G be a simple connected graph with m edges. Then*

$$SDD \geq 2m,$$

with equality if and only if G is regular.

Theorem 2.22. *[37] Let G be a simple connected graph with $n \geq 3$ vertices. Then*

$$SDD \geq 2n - 1.$$

The equality holds if and only if G is isomorphic to path P_n with n vertices.

Theorem 2.23. *[37] Let G be a graph with n vertices and minimum vertex degree δ. Then*

$$SDD \geq n\delta,$$

the equality holds if and only if G is δ-regular graph.

3 Upper bounds for *SDD*

The most important upper bounds obtained for the *SDD* of graphs, in terms of other indices and graph parameters, are listed in this section. Obviously, the bounds that are only in terms of Δ and δ or even the n and m, are not suitable bounds.

Theorem 3.1. *[4] Let G be a connected graph with $m \geq 2$ edges. Then*

$$SDD \leq m\left(\sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}}\right)^2 - 2m.$$

Theorem 3.2. *[4] Let G be a connected graph of order n with $m \geq 2$ edges. Then*

$$SDD \leq 4AG^2 - \frac{m(m-1)\left(\Pi_1^*\right)^{\frac{2}{m}}}{\left(\Pi_2\right)^{\frac{1}{m}}} - 2m, \tag{6}$$

with equality holding if and only if $\frac{d_i}{d_j} + \frac{d_j}{d_i} = \frac{d_k}{d_l} + \frac{d_l}{d_k}$ for any edges $v_iv_j, v_kv_l \in E(G)$.

Theorem 3.3. *[4] Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$SDD \leq n(\Delta_e + \delta_e) - \Delta_e\delta_e R_{-1} - 2m, \tag{7}$$

$$SDD \leq \frac{n^2(\Delta_e + \delta_e)^2}{4\Delta_e\delta_e R_{-1}} - 2m.$$

Equalities hold if and only if G is a regular graph or a semi-regular bipartite graph, or $d_i + d_j = \Delta_e$ or δ_e for any edge $v_iv_j \in E(G)$ $(\Delta_e \neq \delta_e)$.

Theorem 3.4. *[4] Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$SDD \leq n(\Delta_e + \delta_e) - \frac{m^2\delta_e\Delta_e}{M_2} - 2m,$$

$$SDD \leq \frac{n^2 M_2(\Delta_e + \delta_e)^2}{4m^2\Delta_e\delta_e} - 2m,$$

$$SDD \leq \frac{n^2}{R_{-1}} + \frac{(\Delta_e - \delta_e)^2 R_{-1}}{4} - 2m.$$

Equalities hold if and only if G is a regular graph or a semi-regular bipartite graph.

Theorem 3.5. *[11] Let G be a graph of order n, with $m \geq 1$ edges. Then*

$$SDD(G) \leq M_1(G),$$

with equality holding if and only if $G \cong pK_2 \cup (n-2p)K_1$, $(p > 1)$.

Theorem 3.6. *[11] Let G be a graph of order n, with $m \geq 1$ edges. Then*

$$SDD(G) \leq F(G),$$

with equality holding if and only if $G \cong rK_2 \cup (n-2r)K_1, (r \geq 1)$. Moreover, if G is connected and $n > 3$, then $SDD(G) < F(G)$.

Theorem 3.7. *[11] Let G be a graph of order n with m edges and maximum degree δ. Then*

$$SDD \leq \frac{\sigma}{\delta^2} + 2m,$$

with equality holding if and only if G is a regular graph.

Theorem 3.8. *[14] Let G be a graph on n vertices and m edges, with maximum degree Δ and minimum degree δ. Then*

$$SDD \leq \frac{(\sqrt{2}+H)^2 AG^2}{\sqrt{2}mH} - 2m, \tag{8}$$

where $H = \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}}$.

Theorem 3.9. *[15] Let G be a simple connected graph with order n, size m, p pendent vertices, maximum degree Δ and minimum non-pendent vertex degree δ_1. Then*

$$SDD(G) \leq p\left(\frac{\Delta^2+1}{\Delta}\right) + (m-p)\left(\frac{\Delta^2+\delta_1^2}{\Delta\delta_1}\right).$$

Equality holds only if graph is regular and star.

Theorem 3.10. *[15] Let G be a simple connected graph with order n, size m, p pendent vertices, maximum degree Δ and minimum non-pendent vertex degree δ_1. Then*

$$SDD(G) \leq p\left(\frac{\Delta^2+1}{\Delta}\right) + \frac{1}{\delta_1^2}[HM - p(1+\delta_1)^2] - 2(m-p).$$

Equality holds only if graph is regular and star.

Theorem 3.11. *[15] Let G be a simple connected graph with order n, size m, maximum degree Δ and minimum degree δ. Then*

$$SDD(G) \leq \frac{HM}{\delta^2} - 2m.$$

Theorem 3.12. *[22] Let G be a graph of order n with m edges. Then*

$$SDD \leq m\left(n-1+\frac{1}{n-1}\right),$$

with equality if and only if $G \cong S_n$.

Theorem 3.13. *[22] Let G be a graph of order n with maximum degree Δ. Then*

$$SDD \leq \Delta^2 ID,$$

with equality if and only if G is regular.

Theorem 3.14. *[29] Let G be a graph of order n with m edges*

$$SDD \leq 2m(1 + I(G)) - n^2.$$

The equality holds if and only if $d_i = d_j$ for every pair of non-adjacent vertices i and j of G, in particular by d-regular graphs, complete multipartite $K_{p_1,p_2,\dots,p_r}$ graphs and $(n-1,d)$-regular graphs, for $1 \leq d < n-1$.

Theorem 3.15. *[29] Let G be a graph with n vertices, m edges, maximum degree Δ and minimum degree δ. Then*

$$SDD \leq 2m\left[1 + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2m}{n}\right)\right].$$

The equality is attained by all regular graphs.

Theorem 3.16. *[29] Let G be a graph with n vertices and m edges. Then*

$$SDD \leq 2m\left(\frac{Kf(G) + n}{n-1}\right) - n^2,$$

where the equality is attained by K_n and by the complete bipartite graphs $K_{r,n-r}$, for $1 \leq r \leq \lceil \frac{n}{2} \rceil$.

Theorem 3.17. *[37] Let G be a graph with n vertices and minimum vertex degree Δ. Then*

$$SDD \leq n\Delta,$$

the equality holds if and only if G is Δ-regular graph.

4 Bounds for special graphs

In this section, we have gathered the bounds obtained for the SDD of special graphs such as trees, unicyclic graphs, bicyclic graphs, as well as some other bounds for the SDD, obtained by restricting conditions. An important part of these graphs are molecular graphs whose maximum degree of vertices does not exceed 4. Du et al. [6], have presented some upper bounds for the SDD of trees with specific parameters such as matching number, domination number, independence number, number of pendant vertices, segments, diameter and radius are presented. Also, in [27], Mohanappriya et al. have obtained a general expression for the SDD index of transformation networks of a network.

In [3], Ali et al. give the following bounds for the molecular graphs in the term of n and m.

Theorem 4.1. *[3] Let G be a molecular graph of order n with m edges, where $n-1 \leq m \leq 2n$ and $n \geq 5$. Then*

$$SDD \geq n+m,$$

with equality if and only if G is isomorphic to either the path graph P_n or the cycle graph C_n.

Theorem 4.2. *[3] Let G be a molecular graph of order n, with m edges, where $n-1 \leq m \leq 2n$ and $n \geq 5$.*
(1) *If $m+n \equiv 0$ (mod 3) then*

$$SDD(G) \leq 3n + \frac{m}{2},$$

with equality if and only if G contains no vertices of degrees 2 and 3.
(2) *If $m+n \equiv 1$ or 2 (mod 3) then*

$$SDD(G) \leq 3n + \frac{m}{2} - \frac{1}{2}.$$

The equality holds if and only if either G contains no vertex of degree 2 and contains exactly one vertex of degree 3, which is adjacent to three vertices of degree 4, or G contains no vertex of degree 3 and contains exactly one vertex of degree 2, which is adjacent to two vertices of degree 4.

In the next five theorems, bounds for trees, unicyclic graphs and bicyclic graphs are obtained.

Theorem 4.3. *[14] Let T be a tree with n vertices and maximum degree Δ. Suppose T has p pendent edges. Then*

$$SDD \leq \frac{(n-1)\Delta^2 + p}{2}.$$

Theorem 4.4. *[29] Let T be a Tree with n vertices and diameter d. Then*

$$SDD \leq 2(n-1)\left(1 + \frac{3n}{2} - d\right) - n^2,$$

where the equality is attained by the path graph, P_3.

Theorem 4.5. *[29] Let G be a graph with n vertices. Then*
(1) *if G is a tree*

$$SDD(G) \leq n^2 - 2n + 2,$$

(2) *if G is a unicyclic graph*

$$SDD(G) \leq n^2 - 2n + 2 + \frac{2}{n-1},$$

(3) *if G is a bicyclic graph*

$$SDD \leq n^2 - \frac{4}{3}(n+1) + \frac{4}{n-1}.$$

Theorem 4.6. *[37] Let T be a tree with $n \geq 2$ vertices. Then*

$$SDD \leq (n-1)^2 + 1.$$

The equality holds if and only if T is isomorphic to star graph, S_n.

Theorem 4.7. *[37] Let G be a unicyclic connected graph with $n \geq 3$ vertices. Then*

$$SDD \leq \frac{n+1}{n-1} + (n-1)(n-2) + 2.$$

The equality holds if and only if G isomorphic to the graph S_n^+.

Recall that a pendent edge is an edge incident with a vertex of degree one, whereas a path $u_1 u_2 \ldots u_s$ is said to be a pendent path at u_1 if $d_{u_1} \geq 3$, $d_{u_i} = 2$ for $i = 2, 3, \ldots, s-1$, and $d_{u_s} = 1$. A lower bound for the graphs with k pendent paths has been found by Pan et al. in [30].

Theorem 4.8. *[30] If G is a graph with k pendant paths and m edges then it holds that*

$$SDD \geq 2m + \frac{2k}{3}.$$

For positive integer $n \geq 2$, suppose $\mathbb{T}(n)$, $\mathbb{U}(n)$ and $\mathbb{B}(n)$ are the set of trees, unicyclic graphs and bicyclic graphs on $2n$ vertices with a perfect matching, respectively. In [31] and [32], Rajpoot et al. have shown that (except a few special classes of graphs)
[1] for $T \in \mathbb{T}(n)$, $SDD(T) \geq 4n + 1$,
[2] for $G \in \mathbb{U}(n)$, $SDD(G) \geq 4n + 2$,
[3] for $G \in \mathbb{B}(n)$, $SDD(G) \geq 4(n + 1)$.

They also provided the following upper bounds.

Theorem 4.9. *[31] Let $T \in \mathbb{T}(n)$, $n \geq 4$ vertices. Then*

$$SDD(T) \leq \begin{cases} \frac{1}{8}(45n - 26), & n \text{ is even}, \\ \frac{1}{8}(45n - 27), & n \text{ is odd}. \end{cases}$$

Theorem 4.10. *[31] Let $G \in \mathbb{U}(n)$, $n \geq 4$ vertices. Then*

$$SDD(G) \leq \begin{cases} \frac{1}{8}(45n), & n \text{ is even}, \\ \frac{1}{8}(45n - 1), & n \text{ is odd}. \end{cases}$$

Theorem 4.11. *[32] Let $G \in \mathbb{B}(n)$, $n \geq 6$ vertices and G has a maximum degree at most four. Then*

$$SDD(G) \leq \begin{cases} \frac{1}{8}(45n + 26), & n \text{ is even}, \\ \frac{1}{8}(45n + 25), & n \text{ is odd}. \end{cases}$$

Bound	*COR*	*MAE*
Inequality 1	0.996	0.35
Inequality 2	0.995	0.36
Inequality 3	0.986	1.26
Inequality 4	0.999	0.12
Inequality 5	0.986	0.71

Table 1. The best lower bound for SDD.

Bound	*COR*	*MAE*
Inequality 6	0.977	0.35
Inequality 7	0.975	0.36
Inequality 8	0.984	1.26

Table 2. The best upper bound for SDD.

5 Conclusions

The aim of this paper was to collect the obtained bounds for SDD index and compare these bounds. For this purpose, we compared the values of the general bounds (the bounds in sections 2 and 3), for graphs up to 7 vertices (nearly 1000 graphs) with the actual value of SDD and the results are given in the following tables. Table 1. contains the best lower bounds and Table 2. contains the best upper bounds. In these tables, COR is the correlation between SDD and bound and MAE or mean absolute error is the average of all absolute errors. Comparing the data in Table 1. shows that the lower bound obtained by Ghorbani et al. in Theorem 2.13 has the best correlation and the least amount of error. Among the lower bounds, the bound obtained by Ghorbani et al. in Theorem 3.8 has the best correlation with SDD, but considering the mean absolute errors, it seems that the bound obtained by Das et al. in Theorem 3.2 is the best limit.

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