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On the edge energy of some specific graphs

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Abstract. Let $G = (V, E)$ be a simple graph. The energy of G is the sum of absolute values of the eigenvalues of its adjacency matrix $A(G)$. In this paper we consider the edge energy of G (or energy of line of G) which is defined as the absolute values of eigenvalues of edge adjacency matrix of G. We study the edge energy of specific graphs.

Keywords: energy, edge energy, edge adjacency matrix, line graph
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1 Introduction

In this paper, we are concerned with simple finite graphs, without directed, multiple, or weighted edges, and without self-loops. Let G be such a graph, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $A(G)$ be the (0,1)-adjacency matrix of graph G. The characteristic polynomial of G is $det(A(G) - \lambda I)$ and is denoted by $P_G(\lambda)$. The roots of $P_G(\lambda)$ are called the adjacency eigenvalues of G and since $A(G)$ is real and symmetric, the eigenvalues are real numbers. If G has n vertices, then it has n eigenvalues and we denote its eigenvalues in descending order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\lambda_1, \lambda_2, ..., \lambda_s$ be the distinct eigenvalues of G with multiplicity $m_1, m_2, ..., m_s$, respectively. The multiset $Spec(G) = \{(\lambda_1)^{m_1}, (\lambda_2)^{m_2}, ..., (\lambda_s)^{m_s}\}$ of eigenvalues of $A(G)$ is called the adjacency spectrum of G. The energy $E(G)$ of the graph G

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is defined as the sum of the absolute values of its eigenvalues

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Details and more information on graph energy can be found in [11, 15–17, 23, 24]. There are many kinds of graph energies, such as incidence energy [3, 5], Laplacian energy [10], matching energy [8, 19, 20] and Randić energy [2, 4, 22].

The line graph of G is denoted by $L(G)$, the basic properties of line graphs are found in textbooks, e.g., in [18]. The iterated line graphs of G are then defined recursively as $L^2(G) = L(L(G)), L^3(G) = L(L^2(G)), ..., L^k(G) = L(L^{k-1}(G))$. The basic properties of iterated line graph sequences are summarized in the articles [6,7]. Authors in [21] have shown that, if G is a regular graph of order n and of degree $r \geq 3$, then for each $k \geq 2$, $E(L^k(G))$ depends solely on n and r. In particular, $E(L^2(G)) = 2nr(r-2)$. In [14] authors has established relations between the energy of the line graph of a graph G and the energies associated with the Laplacian and signless Laplacian matrices of G.

In this paper we consider the edge energy of a graph (energy of line graph) and compute it for some specific graphs.

2 Main results

In this section we consider the edge energy of a graph (or the energy of the line of a graph) and obtain some of its properties. First we recall the definition of the edge adjacency matrix of a graph. Note that the edge adjacency energy of a graph is just the ordinary energy of the line graph and has studied in detail. For instance see [14, 21].

Definition 1. Let G be a connected graph with edge set $\{e_1, \ldots, e_m\}$. The edge adjacency matrix of G is defined as a square matrix $A_e = A_e(G) = [a_{ij}]$ where $a_{ij} = 0$ if $i = j$ or e_i and e_j are not adjacent, and $a_{ij} = 1$ if edges e_i and e_j are adjacent.

This matrix is symmetric and all its eigenvalues are real. The edge characteristic polynomial of G is $\phi_e(x) = det(A_e - xI)$.

Definition 2. The edge energy of a graph G is denoted by $E_e(G)$ and defined as

$$E_e(G) = \sum_{i=1}^{m} |\mu_i|,$$

where $\mu_1, \ldots, \mu_m$ are eigenvalues of $A_e(G)$.

Here we are interested to obtain edge energy of some specific graphs. First we consider star graphs $K_{1,n}$.

Theorem 2.1. *For every natural n, $E_e(K_{1,n}) = E(K_n)$.*

Proof. We know that the star graph $K_{1,n}$ has n edges. All its edges are adjacent in a vertex (center). The edge adjacency matrix of this graph is

$$A_e(K_{1,n}) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} = J - I,$$

where J is a square matrix whose all its arrays are 1, and I is identity matrix. The eigenvalues of this matrix are the eigenvalues of adjacency matrix of K_n (see [9]). Therefore $E_e(K_{1,n}) = E(K_n)$. □

Theorem 2.2. *For every natural n, $E_e(P_n) = E(P_{n-1})$.*

Proof. The graph path with n vertices, has $n-1$ edges and no cycles. Its edge adjacency matrix is

$$A_e(P_n) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

This matrix is exactly the adjacency matrix of P_{n-1} (see [9]). Therefore we have the result. □

Theorem 2.3. *For every natural $n \geq 3$, $E_e(C_n) = E(C_n)$.*

Proof. The graph cycle with n vertices, has n edges. Its edge adjacency matrix is

$$A_e(C_n) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

This matrix is exactly the adjacency matrix of C_n ([9]) and so we have the result. □

Now we shall obtain the edge energy of some another graphs. Here we investigate the complete bipartite graphs $K_{m,n}$.

Lemma 2.4. *(i) The edge characteristic polynomial of $K_{m,n}$ is*

$$(x+2)^{(m-1)(n-1)}(x-(m+n-2))(x+2-n)^{m-1}(x+2-m)^{n-1}.$$

(ii) $E_e(K_{m,n}) = 4(m-1)(n-1)$.

Proof. (i) We can see that the edge adjacency matrix of $K_{m,n}$ is $mn \times mn$ matrix

$$A_e(K_{m,n}) = \begin{pmatrix} J_n - I_n & I_n & I_n \cdots & I_n \\ I_n & J_n - I_n & I_n \cdots & I_n \\ \vdots & \vdots & \vdots \;\; \vdots & \\ I_n & I_n & I_n \cdots & J_n - I_n \end{pmatrix}.$$

With simple computation,

$$\begin{aligned} \phi_e(x) &= det(A_e(K_{m,n}) - xI) \\ &= (x+2)^{(m-1)(n-1)}(x-(m+n-2))(x+2-n)^{m-1}(x+2-m)^{n-1}. \end{aligned}$$

(ii) It follows from Part (i). □

Now, we consider two families of graphs and obtain their edge energy. The friendship (or Dutch-Windmill) graph F_n is a graph that can be constructed by coalescence n copies of the cycle graph C_3 of length 3 with a common vertex. The Friendship theorem of Paul Erdös, Alfred Rényi and Vera T. Sós [12], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. The Figure 1. shows some examples of friendship graphs. Let to obtain the energy of F_n. First we need

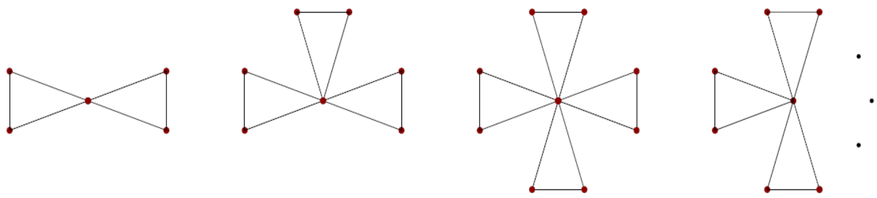

Figure 1. Friendship graphs F_2, F_3, F_4 and F_n, respectively.

the following theorem:

Theorem 2.5. [1]

(i) The characteristic polynomial of F_n is

$$P_{F_n}(x) = (x+1)(x^2-1)^{n-1}(x^2-x-2n).$$

(ii) The spectrum of friendship graph F_n is

$$Spec(F_n) = \Big\{ (\frac{1}{2} - \frac{1}{2}\sqrt{1+8n})^1, (-1)^n, (1)^{n-1}, (\frac{1}{2} + \frac{1}{2}\sqrt{1+8n})^1 \Big\}.$$

The following corollary is an immediate consequence of Theorem 2.5:

Corollary 2.6. *The energy of friendship graph F_n is*

$$E(F_n) = \sqrt{1+8n} + 2n - 1.$$

To obtain the edge energy of friendship graphs, we consider two following matrices:

$$A = \begin{pmatrix} 0\,1\,1 \\ 1\,0\,1 \\ 1\,1\,0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1\,0\,1 \\ 0\,0\,0 \\ 1\,0\,1 \end{pmatrix}.$$

It is easy to see that the edge adjacency matrix of F_n is $3n \times 3n$ matrix in the following lemma:

Lemma 2.7. *The edge adjacency matrix of friendship graph F_n is*

$$A_e(F_n) = \begin{pmatrix} A & B & B \cdots & B \\ B & A & B \cdots & B \\ \vdots & \vdots & \vdots \ddots & \vdots \\ B & B & B \cdots & A \end{pmatrix}.$$

Theorem 2.8. *(i) The edge characteristic polynomial of F_n is*

$$(x^2 - (2n-1)x - 2)(x-1)^{n-1}(x+2)^{n-1}(x+1)^n.$$

(ii) The edge energy of friendship graphs is $E_e(F_n) = 4n - 3 + \sqrt{(2n-1)^2 + 8}$.

Proof. (i) Using Lemma 2.7 and simple computation we have the result.

(ii) It follows from Part (i). □

Let us to consider book graphs. The n-book graph B_n can be constructed by joining n copies of the cycle graph C_4 with a common edge $\{u, v\}$, see Figure 2.

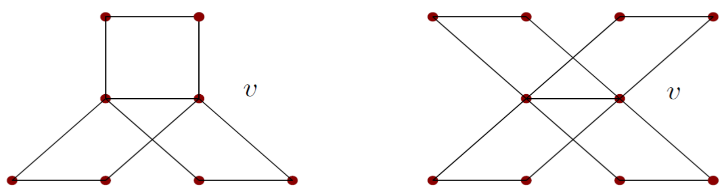

Figure 2. The book graphs B_3 and B_4, respectively.

The following theorem gives the edge characteristic polynomial and edge energy of book graphs.

Theorem 2.9. *(i) The edge characteristic polynomial of B_n is*

$$x(x-(n-1))(x-(n+1))(x-1)^{n-1}(x+2)^n(x+1)^{n-1}.$$

(ii) The edge energy of book graph is $E_e(B_n) = 6n - 2$.

Proof. (i) Consider the following $n \times (n+1)$ matrix

$$A = \begin{pmatrix} 1\,1\,0\,0 \cdots 0 \\ 1\,0\,1\,0 \cdots 0 \\ \vdots\,\vdots\,\vdots\,\vdots\; \vdots\; \vdots \\ 1\,0\,0\,0 \cdots 1 \end{pmatrix}.$$

It is easy to see that the edge adjacency matrix of B_n is the following $(3n+1)\times(3n+1)$ matrix:

$$A_e(B_n) = \begin{pmatrix} J-I & A & 0 \\ A^t & 0 & A^t \\ 0 & A & J-I \end{pmatrix}.$$

With simple computation, we see that

$$\begin{aligned} \phi_e(x) &= det(A_e(B_n) - xI) \\ &= x(x-(n-1))(x-(n+1))(x-1)^{n-1}(x+2)^n(x+1)^{n-1}. \end{aligned}$$

(ii) It follows from Part (i). □

In the end of this paper, we present the edge adjacency matrix of two another kind of graphs. For this purpose, we need the following matrix

$$B = \begin{pmatrix} 1\,1\,0\,0 \cdots 0 \\ 0\,1\,1\,0 \cdots 0 \\ \vdots\,\vdots\,\vdots\,\vdots\; \vdots\; \vdots \\ 1\,0\,0\,0 \cdots 1 \end{pmatrix}.$$

For two graphs $G = (V, E)$ and $H = (W, F)$, the corona $G \circ H$ is the graph arising from the disjoint union of G with $|V|$ copies of H, by adding edges between the ith vertex of G and all vertices of ith copy of H ([13]).

It is not difficult to see that the edge adjacency matrices of wheel graphs $W_{n+1} = C_n + K_1$ and graphs $C_n \circ K_1$ are in the form stated in the following theorem.

Theorem 2.10. *(i) The edge adjacency matrix of wheel graphs W_{n+1} is the following $2n \times 2n$ matrix:*

$$A_e(W_n) = \begin{pmatrix} A_e(C_n) & B \\ B^t & J-I \end{pmatrix}.$$

(ii) The edge adjacency matrix of graphs $C_n \circ K_1$ is the following $2n \times 2n$ matrix:

$$A_e(C_n \circ K_1) = \begin{pmatrix} A_e(C_n) & B \\ B^t & 0 \end{pmatrix}.$$

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