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Research Paper

Automorphism group of quasi-abelian semi-Cayley graphs

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Abstract. Let G be a group and R, L, S be subsets of G such that $R = R^{-1}$, $L = L^{-1}$ and $1 \notin R \cup L$. The undirected graph $(G; R, L, S)$ with vertex set union of $G_1 = \{g_1 \mid g \in G\}$ and $G_2 = \{g_2 \mid g \in G\}$, and edge set the union of $\{\{g_1, (gr)_1\} \mid g \in G, r \in R\}$, $\{\{g_2, (gl)_2\} \mid g \in G, l \in L\}$ and $\{\{g_1, (gs)_2\} \mid g \in G, s \in S\}$ is called semi-Cayley graph over G. We say that $(G; R, L, S)$ is quasi-abelian if R, L and S are a union of conjugacy classes of G. In this paper, we study the automorphism group of quasi-abelian semi-Cayley graphs.

Keywords: semi-Cayley graph, quasi-abelian semi-Cayley graph, automorphism of graph
Mathematics Subject Classification (2010): 05E30.

1 Introduction

A graph Γ is called a *semi-Cayley graph* over a group G if (Γ), the automorphism group of Γ, contains a semiregular subgroup isomorphic to G with two orbits (of equal size). Resmini and Jungnickel determined the structure representation of semi-Cayley graphs in [9]. They proved that a graph Γ is a semi-Cayley graph over a group G if there exist subsets R, L and S of G such that $R = R^{-1}$, $L = L^{-1}$ where $1 \notin R \cup L$ such that Γ is isomorphic to the graph $(G; R, L, S)$, where $(G; R, L, S)$ is a graph with vertex the union of the right part $G_1 = \{g_1 \mid g \in G\}$ and the left part $G_2 = \{g_2 \mid g \in G\}$, and its edge set is the union of $\{\{g_1, (gr)_1\} \mid g \in G, r \in R\}$, $\{\{g_2, (gl)_2\} \mid g \in G, l \in L\}$ and $\{\{g_1, (gs)_2\} \mid g \in G, s \in S\}$. It is easy to see that

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$R_G = \{\rho_g \mid g \in G\}$, where $\rho_g : G_1 \cup G_2 \to G_1 \cup G_2$ and $x_i^{\rho_g} = (xg)_i$, $i = 1,2$, is a semiregular subgroup of automorphism group of $(G; R, L, S)$ isomorphic to G with two orbits G_1 and G_2. The semi-Cayley graph $(G; R, L, S)$ is called a *quasi-abelian* semi-Cayley graph over G if R, L and S are a union of conjugacy classes of G [4]. Clearly, every semi-Cayley graph over an abelian group G is a quasi-abelian semi-Cayley graph over G.

The class of semi-Cayley graphs contains many families of graphs, such as the Cayley graphs on a finite group having a subgroup of index 2 and generalized Petersen graphs, which have been an object of interest for many years, see for example [1–3, 5, 6, 8]. Very recently, some graph theoretic properties of quasi-abelian semi-Cayley graphs are studied [4, 10, 11]. In this paper, we study the automorphism group of these graphs.

2 Results and discussion

Let Γ be a semi-Cayley graph over a group G. Then for all $g \in G$, we define the following maps on $V(\Gamma)$:

$$\begin{aligned} \rho_g &: V(\Gamma) \to V(\Gamma);\ \ x_i^{\rho_g} = (xg)_i, \\ \psi_g &: V(\Gamma) \to V(\Gamma);\ \ x_i^{\psi_g} = (gx)_i, \\ \theta_g &: V(\Gamma) \to V(\Gamma);\ \ x_i^{\theta_g} = (g^{-1}xg)_i. \end{aligned}$$

Let $R_G = \{\rho_g \mid g \in G\}$, $L_G := \{\psi_g \mid g \in G\}$ and $_G = \{\theta_g \mid g \in G\}$. Clearly $R_G, L_G, _G$ are bijections on $V(\Gamma)$. Furthermore, $R_G L_G = R_G {}_G$, since for all $g, h \in G$, we have $\rho_g \psi_h = \rho_{gh} \theta_{h^{-1}}$. Also $R_G \leq (\Gamma)$ and if Γ is quasi-abelian, then $_G \leq (\Gamma)$. In particular, if G is abelian, then $L_G = R_G \leq (\Gamma)$ and $_G$ is the identity subgroup of (Γ). In the following result which is a direct consequence of [4, Corollary 2.3], we gather some equivalent conditions for a semi-Cayley graphs to be quasi-abelian..

Proposition 2.1. *Let $\Gamma = (G; R, L, S)$ be a semi-Cayley graph over group G. Then the following are equivalent*

(1) Γ *is quasi-abelian.*

(2) $L_G \leq (\Gamma)$.

(3) $_G \leq (\Gamma)$.

(4) $R_G L_G \leq (\Gamma)$.

(5) $R_G Inn_G \leq (\Gamma)$.

Let $\xi_G : G_1 \cup G_2 \to G_1 \cup G_2$ be a map by the rule $x_i^{\xi_G} = (x^{-1})_i$. In the following lemma, we determine semi-Cayley graphs that their automorphism group contains ξ_G.

Lemma 2.2. *Let $\Gamma = (G; R, L, S)$ be a semi-Cayley graph over G. Then $\xi_G \in (\Gamma)$ if and only if Γ is quasi-abelian and $S = S^{-1}$.*

Proof. Suppose $\xi_G \in (\Gamma)$. Let $s \in S$. Then $\{1_1, s_2\} \in E(\Gamma)$. Since $\xi_G \in (\Gamma)$, we have $\{1_1, (s^{-1})_2\} \in E(\Gamma)$ which means that $s^{-1} \in S$. This proves that $S = S^{-1}$. Now we prove that Γ is quasi-abelian. Let $R = T_{11}$, $L = T_{22}$ and $S = T_{12}$. Since Γ is undirected and $S = S^{-1}$, T_{ij} is inverse-closed for all i, j. Let $t \in T_{ij}$ for some i, j and $g \in G$. Then $\{1_i, t_j\} \in E(\Gamma)$ and moreover,

$$\begin{aligned}\{1_i, t_j\} \in E(\Gamma) &\Leftrightarrow \{g_i, (tg)_j\} \in E(\Gamma)\\ &\Leftrightarrow \{g_i^{\xi_G}, (tg)_j^{\xi_G}\} \in E(\Gamma)\\ &\Leftrightarrow \{(g^{-1})_i, (g^{-1}t^{-1})_j\} \in E(\Gamma),\end{aligned}$$

which implies that $g^{-1}tg \in T_{ij}^{-1} = T_{ij}$. This means that Γ is quasi-abelian.

Conversely, suppose that Γ is quasi-abelian and $S = S^{-1}$. Then, by Corollary 2.1, $_G \leq (\Gamma)$. Since Γ is undirected $R = R^{-1}$ and $L = L^{-1}$. So $\{x_i, y_j\} \in E(\Gamma)$ if and only if $\{y_i, x_j\} \in E(\Gamma)$. On the other hand,

$$\begin{aligned}\{y_i, x_j\} \in E(\Gamma) &\Leftrightarrow \{y_i^{\theta_y}, x_j^{\theta_y}\} \in E(\Gamma)\\ &\Leftrightarrow \{y_i, (y^{-1}xy)_j\} \in E(\Gamma)\\ &\Leftrightarrow \{y_i^{\rho_{y^{-1}x^{-1}}}, (y^{-1}xy)_j^{\rho_{y^{-1}x^{-1}}}\} \in E(\Gamma)\\ &\Leftrightarrow \{(x^{-1})_i, (y^{-1})_j\} \in E(\Gamma),\end{aligned}$$

which proves that $\xi_G \in (\Gamma)$. □

Recall that a semi-Cayley graph $(G; R, L, S)$ is called *one-matching* over G if $S = \{1\}$ [8]. The following result is a direct consequence of Lemma 2.2.

Corollary 2.3. *Let Γ be a one-matching semi-Cayley graph over a group G. Then Γ is quasi-abelian if and only if $\xi_G \in (\Gamma)$.*

By Proposition 2.1, if Γ is a quasi-abelian semi-Cayley graph over a group G, then $R_G L_G$ is a subgroup of (Γ). In the following theorem, we determine quasi-abelian semi-Cayley graphs with as small as possible automorphism group in some sense.

Theorem 2.4. *Let $\Gamma = (G; R, L, S)$ be a semi-Cayley graph over a finite group G and $S = S^{-1}$. Then $(\Gamma) = R_G L_G$ if and only if G is an elementary abelian 2-group and $(\Gamma) = R_G$.*

Proof. Let $(\Gamma) = R_G L_G$. Then, by Corollary 2.1, Γ is quasi-abelian. Hence, by Lemma 2.2, $\xi_G \in (\Gamma)$. On the other hand, R_G and L_G commute each other and so R_G is a normal subgroup of (Γ). Now since ξ_G fixes 1_1, [3, Proposition 2(2)] implies that there exists $\sigma \in (G)$ such that for all $x \in G$, $x^\sigma = x^{-1}$, which means that G is abelian. Hence $R_G = L_G$ and $(\Gamma) = R_G$. On the other hand, $\xi_G \in (\Gamma)$ implies that there exist $x, y \in G$ such that $\xi_G = \rho_x \psi_y$. Again since ξ_G fixes 1_1, we have $y = x^{-1}$. So for all $g \in G$,

$$(g^{-1})_1 = g_1^{\xi_G} = g_1^{\rho_x \psi_{x^{-1}}} = (x^{-1}gx)_1 = g_1,$$

which implies that G is a elementary abelian 2-group. This proves one direction. Conversely, suppose that G is an elementary abelian 2-group and $(\Gamma) = R_G$. Then $R_G = L_G$. This means that $(\Gamma) = R_G L_G$ as desired. □

By Theorem 2.4, it is a natural question that for which elementary abelian 2-group G there exists a semi-Cayley graph Γ over G such that $(\Gamma) = R_G$? Let $R(G) = \{r_g \mid g \in G\}$, where $r_g : G \to G$ is the map by the rule $x \mapsto xg$. Then $R(G)$ is a regular subgroup of any Cayley graph over G. To attack to the problem, we need the following result which construct a connection between Cayley graphs and semi-Cayley graphs. One can find the proof of this result in [3, proof of Lemma 4.1], but we give it for completeness.

Lemma 2.5. *Let T be a non-empty inverse-closed subset of a finite group G not containing 1, $\Sigma = (G,T)$ be a Cayley graph over G and $\Gamma = (G;R,L,S)$, where $R = T$, $L = \varnothing$ and $S = \{1\}$. If $(\Sigma) = R(G)$, then $(\Gamma) = R_G$.*

Proof. We define $\psi : R(G) \to (\Gamma)$, where $r_g^{\psi} = \rho_g$. Clearly ψ is well-defined and $1-1$. Since $r_{g_1}r_{g_2} = r_{g_1g_2}$ and $\rho_{g_1}\rho_{g_2} = \rho_{g_1g_2}$ for all $g_1, g_2 \in G$, ψ is a group homomorphism. Now we show that ψ is onto. Let $\varphi \in (\Gamma)$. We claim that φ fixes G_1 setwise. To see this, suppose towards a contradiction that $x_1^{\varphi} = y_2$ for some $x, y \in G$. Since for all $g \in G$ the only adjacent vertex to g_2 is g_1, we conclude that $x_2^{\varphi} = y_1$. Hence for all $t \in T$, we have $(tx)_1^{\varphi} = (y)_1$, which implies that $(tx)_1 = (x)_1$. So for all $t \in T$, we have $tx = x$ which means that $t = 1$, a contradiction. So our claim is true and the restriction of φ to G_1 induces an automorphism of Σ. Furthermore, we may assume that for all $g \in G$, $(g)_1^{\varphi} = (g^{\sigma})_1$ for some $\sigma \in (\Sigma)$. Let $g \in G$. Then $g_2^{\varphi} \in G_2$. Since $(g)_1$ is adjacent to g_2, we conclude that $(g^{\sigma})_2 = (g_2)^{\varphi}$. Hence for all $g \in G$ and $i \in \{1,2\}$, we have $(g)_i^{\varphi} = (g^{\sigma})_i$, which means that $\sigma^{\psi} = \varphi$. This shows that ψ is onto and so $R(G) \cong (\Gamma)$, which implies that $(\Gamma) = R_G$. □

Now we are ready to answer to the above question.

Theorem 2.6. *For every finite elementary abelian 2-group G there exists a semi-Cayley graph over G such that $(\Gamma) = R_G$.*

Proof. Let $G = \langle a_1 \rangle \times \ldots \times \langle a_n \rangle \cong \mathbb{Z}_2^n$, $n \geq 1$, be an elementary abelian 2-group. Then one can check that in all of the following cases we have $((G;R,L,S)) = R_G$.

If $n = 1$, put $R = \{a_1\}$, $L = \varnothing$ and $S = \{1\}$.

If $n = 2$, put $R = \{a_1, a_2\}$, $L = \{a_2\}$ and $S = \{1\}$.

If $n = 3$, put $R = \{a_3\}$, $L = \{a_2, a_3\}$ and $S = \{1, a_1, a_2, a_3\}$.

If $n = 4$, then put $R = \varnothing$, $L = \{a_3, a_4\}$ and $S = \{1, a_1, a_2, a_3, a_4, a_1a_2, a_2a_4, a_1a_3a_4\}$.

Now let $n \geq 5$. Then, by [7], there exists an undirected Cayley graph $\Sigma = (G,T)$ over G such that $(\Sigma) = R(G)$. Let $\Gamma = (G;R,L,S)$, where $R = T$, $L = \varnothing$ and $S = \{1\}$. By Lemma 2.5, $(\Gamma) = R_G$, which completes the proof. □

By Lemma 2.1, if $\Gamma = (G;R,L,S)$ is quasi-abelian and $S = S^{-1}$, then $R_G L_G \langle \xi_G \rangle \leq (\Gamma)$. Hence it is an interesting question that how far $R_G L_G \langle \xi_G \rangle$ is from (Γ)? In the rest of paper, we determine quasi-abelian semi-Cayley graphs that $R_G L_G \langle \xi_G \rangle$ is equal to their automorphism. We need the following lemma.

Lemma 2.7. *Let G be a group. Then $(G \times G)/C \cong R_G L_G$, where $C = \{(x,x) \mid x \in Z(G)\}$.*

Proof. Define $\varphi : G \times G \to R_G L_G$ by the rule $(x,y)^\varphi = \rho_x \psi_y$. Then it is easy to see that φ is a group epimorphism with kernel C. □

Now we are ready to determine quasi-abelian semi-Cayley graphs that $R_G L_G \langle \xi_G \rangle$ is equal to their automorphism group.

Theorem 2.8. *Let $\Gamma = (G; R, L, S)$ be an undirected quasi-abelian semi-Cayley graph over a finite group G and $S = S^{-1}$. Then $(\Gamma)_{1_1} \geq_G \times \langle \xi_G \rangle$. Furthermore,*

(a) If Γ is not vertex-transitive, then $(\Gamma) = R_G L_G \langle \xi_G \rangle$ if and only if $(\Gamma)_{1_1} =_G \times \langle \xi_G \rangle$.

(b) If Γ is vertex-transitive, then $(\Gamma) = R_G L_G \langle \xi_G \rangle$ if and only if $|(\Gamma)_{1_1}| = |G : Z(G)|$ and G is non-abelian.

Proof. Put $A = (\Gamma)$. Since Γ is quasi-abelian and $S = S^{-1}$, $\langle \xi_G \rangle$ and $_G$ are subgroups of A and each of them fixes 1_1. Furthermore, for all $\theta_g \in_G$ we have $\xi_G \theta_g = \theta_g \xi_G$. On the other hand, $_G \cap \langle \xi_G \rangle$ is the trivial subgroup of Γ. This proves that $A_{1_1} \geq_G \times \langle \xi_G \rangle$.

Let Γ is not vertex-transitive. We claim that A fixes G_1 setwise. To see this, suppose by contrary that there exists $\varphi \in A$ and $g, h \in G$ such that $g_1^\varphi = h_2$. Then $\langle R_G, \varphi \rangle$ is a transitive subgroup of A, which is a contradiction. Hence our claim is true and A acts on $G \times \{1\}$. Since R_G acts transitively on G_1, we have $A = A_{1_1} R_G$. In particular, $|A| = |G||A_{1_1}|$.

Let $A = R_G L_G \langle \xi_G \rangle$. Then $R_G L_G = R_{GG}$ implies that $A = R_{GG} \langle \xi_G \rangle$. Since $R_G \cap_G \langle \xi_G \rangle = \{1\}$, $|A| = |G||_G \langle \xi \rangle|$. Hence $|_G \langle \xi \rangle| = |A_{1_1}|$, which implies that $A_{1_1} =_G \times \langle \xi_G \rangle$. The converse follows from the equalities $R_G L_G = R_{GG}$ and $A = A_{1_1} R_G$. This proves (a).

Finally, we prove (b). Let Γ be vertex-transitive. Then $|A| = |V(\Gamma)||A_{1_1}| = 2|G||A_{1_1}|$.

Let $A = R_G L_G \langle \xi_G \rangle$. If G is abelian, then $R_G = L_G$ and so $A = R_G \langle \xi_G \rangle$. Hence $|A| = |G|$ or $2|G|$ whenever G is elementary abelian 2-group or not, respectively. The first case is impossible and the later implies that $|A_{1_1}| = 1$ which implies that $\xi_G = 1$ i.e G is elementary abelian 2-group, a contradiction. Hence G is non-abelian. If $\xi_G \in R_G L_G$, then $\xi_G = \rho_x \psi_y$ for some $x, y \in G$. Hence $1_1 = 1_1^{\xi_G} = 1_1^{\rho_x \psi_y} = (yx)_1$ which implies that $y = x^{-1}$. Now $(x^{-1})_1 = (x)_1^{\xi_G} = (x)_1^{\rho_x \psi_{x^{-1}}} = x_1$ which implies that $x = x^{-1} = y$. Hence $\xi_G = \rho_x \psi_x$, where $x^2 = 1$. So $\rho_x \xi_G = \rho_x^2 \psi_x = \psi_x$. Thus for all $g \in G$ we have $(xg)_1 = g_1^{\psi_x} = g_1^{\rho_x \xi_G} = (xg^{-1})_1$ which implies that $g^{-1} = g$. This means that G is abelian, a contradiction. Hence $\xi \notin R_G L_G$ and so $|A| = |R_G L_G||\langle \xi_G \rangle| = 2\frac{|G|^2}{|Z(G)|}$ by Lemma 2.7. Hence $|A_{1_1}| = |G : Z(G)|$.

Conversely, suppose that $|A_{1_1}| = |G : Z(G)|$. Since G is non-abelian $\xi_G \neq 1$ and $|R_G L_G \langle \xi_G \rangle| = |R_G L_G||\langle \xi_G \rangle| = 2\frac{|G|^2}{|Z(G)|}$, where the last equality obtained from Lemma 2.7. Hence $|R_G L_G \langle \xi_G \rangle| = 2|G||A_{1_1}|$ which implies that $A = R_G L_G \langle \xi_G \rangle$. This completes the proof. □

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