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Research paper

Determination of the Maximum Dynamic Range of Sinusoidal Frequencies in A Wireless Sensor Network with Low Sampling Rate

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Abstract

Background and Objectives: Subsampling methods allow sampling signals at rates much lower than Nyquist rate by using low-cost and low-power analog-to-digital converters (ADC). These methods are important for systems such as sensor networks that the cost and power consumption of sensors are the core issue in them. The Chinese remainder theorem (CRT) reconstructs a large integer (input frequency) from its multiple remainders (aliased or under-sampled frequencies), which are produced from under-sampling or integer division by several smaller positive integers. Sampling frequencies can be reduced by approaches based on CRT.

Methods: The largest dynamic range of a generalized Chinese remainder theorem for two integers (input frequencies) has already been introduced in previous works. This is equivalent to determine the largest possible range of the frequencies for a sinusoidal waveform with two frequencies which the frequencies of the signal can be reconstructed uniquely by very low sampling frequencies. In this study, the largest dynamic range of CRT for any number of integers (any number of frequencies in a sinusoidal waveform) is proposed. It is also shown that the previous largest dynamic range for two frequencies in a waveform is a special case of our proposed procedure.

Results: A procedure for multiple frequencies detection from reminders (under-sampled frequencies) is proposed and maximum tolerable noises of under-sampled frequencies for unique detection is obtained. The numerical examples show that the proposed approach, in some cases, can gain 11.5 times higher dynamic range than the conventional methods for a multi-sensor under-sampling system.

Conclusion: Other studies introduced the largest dynamic range for the unique reconstruction of two frequencies by CRT. In this study, the largest dynamic ranges for any number of frequencies are investigated. Moreover, tolerable noise is also considered.

Introduction

The Chinese Remainder Theorem (CRT) is a well-known research topic, which reconstructs a large positive integer from its remainders [1]-[3]. Nowadays, CRT is widely used in different applications including signal processing, image processing, etc. In our previous works, the high range of frequency is estimated by sensors with a very low sampling rate [1], [2]. In [4], the CRT algorithm is used to achieve range estimation of multiple targets in a pulse Doppler radar when the measured ranges are overlapped with noise error. An approach based on CRT is introduced in [5] to estimate frequencies when a signal is under sampled by multiple under-sampling frequencies.

In [6], the statistical model of CRT-based multiple parameter estimation is investigated, and two approaches are introduced to address the problems of ambiguity resolution in parameter estimation.

A method based on CRT is introduced in [7] to estimate the direction of arrival (DOA) of the signal. This algorithm has less complexity with similar precision in comparison with other algorithms for DOA estimation.

In [8], CRT and non-orthogonal multiple access (NOMA) techniques are introduced for unmanned aerial vehicle (UAV) relay networks to improve communication between transmitter and receiver.

A combination of CRT with Haar Wavelet Transform was proposed as a watermarking technique in [9] to hide information.

The Haar Wavelet Transform has been used for imperceptibility, and CRT provides the security of the watermarked image. A reversible sketch data structure based on CRT was proposed in [10] to compress and fuse big data network traffic. In [11], a novel CRT-based conditional privacy was introduced to keep an authentication scheme for securing vehicular authentication.

In this work, the CRT can help the trusted authorities to generate and broadcast group keys to the network vehicles.

In [12], the authors proposed a multiple secret image-sharing scheme by CRT and Boolean exclusive-OR operation.

A robust and secure data-hiding method in the Tchebichef domain is presented based on CRT [13]. The efficiency of the algorithm was confirmed by implementing the algorithm over different images.

Power efficiency is one of the critical design factors in wireless sensor network systems. In such systems, it is possible to digitalize received analog signal by sensors with very low frequencies and use CRT for manipulation and reconstruction of the frequencies of the main signal. In [14] a low-frequency power efficient digital signal processing architecture for mathematic operations based on CRT was designed and implemented.

A packet forwarding scheme based on CRT was developed for wireless sensor networks in [15]. The advantages of this scheme are energy efficiency, low computational complexity and high reliability.

In [16], an approach based on the frequency domain sparse common support and CRT was developed for frequency determination of multiple sinusoidal signals when the sampling rate even less than Nyquist rate. Authors in [17] proposed an approach to reconstruct the multiple frequencies of a sinusoidal waveform from aliased frequencies by the CRT approach.

In all these researches the dynamic range for unambiguously reconstruction integers (e.g. frequencies), which are divided by a set of modules (e.g. sampling frequencies) from their remainders (e.g. aliased frequencies) is important.

The higher dynamic range for a set of modules means the possibility to reconstruct the larger range of integers un-ambiguously by remainder of integers from those modules. Thus, any improvement in the dynamic range will lead to more efficient schemes in many applications [3].

The dynamic range for the unique determination of an integer (frequency) N_1 with modules (sampling frequencies) $\Gamma = \{m_1, m_2, \ldots, m_\gamma\}$ is the least common multiple (lcm) of modules i.e., $d = lcm(m_1, m_2, \ldots, m_\gamma)$ [5], [18]. A dynamic range for the unique determination of two integers (frequencies) N_1 and N_2 can be obtained as $d = \min_{I_1, I_2}\{max\{I_1, I_2\}\}$ where $I_1 \cup I_2 = \Gamma$ [19]. The first generic dynamic range for reconstruction of multiple integers (more than two integers (frequencies)) from their modules was introduced in [20].

A sharpened dynamic range for ρ integers ($\rho = 1,2,\ldots$) was presented as $d = \min_{I_1,\ldots,I_\rho}\{max\{I_1, \ldots, I_\rho\}\}$ where $\bigcup_{i=1}^{\rho} I_i = \Gamma$ in [19]. Dynamic ranges for multiple integers when there are conditions over integers are presented in in [21], [22].

The largest dynamic range for two integers is obtained as $d = \min_{I_1, I_2}\{I_1 + I_2\}$ where $I_1 \cup I_2 = \Gamma$ in [23] and the maximum tolerable error for two integers was discussed in [24] and it is applied in [25] for the direction of arrival (DOA) of two sources and in [26], [27] was used for secret image sharing by the modular operation.

Most of the previous studies discussed the un-ambiguous dynamic range for two integers (frequencies) or assumed conditions for integers (frequencies) [21], [22], [28] that will be discussed with details in the background section while we present a close form relationship of the largest dynamic range for multiple integers (frequencies) without condition on them. Furthermore, we show that the largest dynamic range for reconstruction of two integers (frequencies) is a special case of our work.

The presentation is organized as follows. Related works with theoretical background is discussed in Background section.

A proposition for finding the maximum possible range for unique reconstruction of any number of input frequencies from under-sampled frequencies is introduced in Proposed Approach section. Furthermore, the proposed proposition is specified for two and three input frequencies in corollaries, the maximum tolerable noise for the maximum possible range is obtained and a procedure for reconstruction is also introduced in the Proposed Approach section.

Different numeric examples to verify the effectiveness of the proposed approaches are introduced in the Simulation Results section.

Finally, the work is concluded in the Conclusion section.

Background

Consider a complex waveform without noise as follows [23]:

$$x(t) = \sum_{l=1}^{\rho} A_l e^{i(2\pi F_l t+\varphi_l)} + w(t) \tag{1}$$

where A_l's are unknown nonzero complex coefficients and F_l's ; $1 \le l \le \rho$are multiple unknown frequencies in Hz that should be determined. The $w(t)$ is additive white Gaussian noise.

Consider γ sensors in a wireless sensor network with $\Gamma = \{f_{s1}, f_{s2}, \ldots, f_{s\gamma}\}$; $\gamma \ge 2$ sampling rates as Fig. 1 in which all may be much less than the unknown frequencies i.e. $f_{si} << F_l\,; i = 1, \ldots, \gamma\,; l = 1, \ldots, \rho$[1], [2].

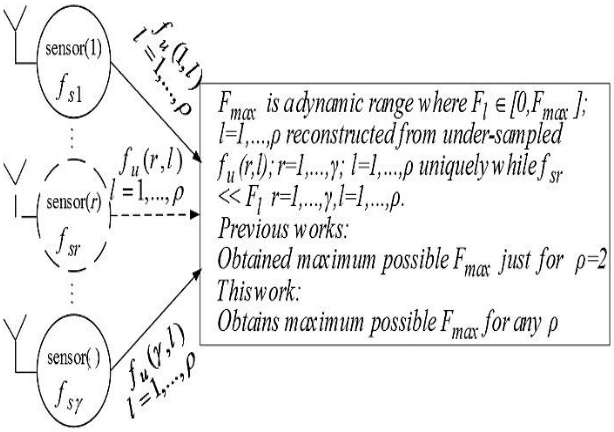

Fig. 1: A multi-sensor system for the determination of frequencies by information fusion from sensors.

Assume these sampling frequencies are co-prime i.e. $M = lcm(f_{s1}, f_{s2}, \ldots, f_{s\gamma}) = f_{s1} f_{s2} \ldots f_{s\gamma}$ and without loss of generality assume $f_{s1} < f_{s2} < \ldots < f_{s\gamma}$that lcm is the least common multiplier.

Then, the multiple under-sampled waveforms by sampling frequency $f_{sr}\,; r = 1, \ldots, \gamma$ are given by [23]:

$$x_{f_{sr}}(n) = \sum_{l=1}^{\rho} A_l e^{2\pi j F_l n / f_{sr}}\,; \quad n \in \mathbb{Z} \tag{2}$$

Using the f_r-point discrete Fourier transform (DFT) to $x_{f_{sr}}(n)$, relation (2) can be written as:

$$DFT_{f_{sr}}\left(x_{f_{sr}}(n)\right)[k] = \sum_{l=1}^{\rho} A_l \delta\left(k - f_{u(l,r)}\right), \quad 1 \le r \le \gamma \tag{3}$$

where $\delta(k)$ is equal to 1 when $k = 0$ and others $\delta(k) = 0$. The $f_{u(l,r)}$is remainder (under-sampled frequency) of F_l with module (sampling frequency) f_{sr} i.e. $f_{u(l,r)} = F_l \bmod f_{sr}$.

Thus, following under-sampled frequencies set $S_r(F_1, \ldots, F_\rho)$ can be written.

$$S_r(F_1, \ldots, F_\rho) = \bigcup_{l=1}^{\rho} \{f_{u(l,r)}\}, \quad r = 1, \ldots, \gamma \tag{4}$$

Consider F_{max} be an upper bound of input frequencies when all input frequencies less than F_{max} (i.e., $F_l \le F_{max}$ $, l = 1, \ldots \rho$) can be uniquely reconstructed from their remainders. Some works have been done to determine F_{max} for unambiguous reconstruction of multiple integers (multiple frequencies) from their remainders sets where we briefly review them in the sequel.

Proposition 1 [29], [30]: A large dynamic range (F_{max}) for unique determination $F_l, l = 1, \ldots \rho$ when under-sampled with $f_{si}, i = 1, \ldots, \gamma$ is $F_{max} = max(\upsilon, f_{s\gamma})$ where $\upsilon = \min_{1 \le r_1 \le \ldots r_\eta \le \gamma} lcm\{f_{s(r_1)}, \ldots, f_{s(r_\eta)}\}$ that $\gamma = \eta\rho + \theta$ for some $0 \le \theta < \rho$.

A proposed majority method for the determination multiple integers from their moduli introduced in [19] as follows:

Proposition 2 [19]: A large dynamic range for multiple integers (multiple frequencies) is $F_{max} = \min_{I_1 \cup \ldots \cup I_\rho = \Gamma} max\{\prod_{f_{si} \in I_1} f_{si}, \ldots, \prod_{f_{si} \in I_\rho} f_{si}\}$ where $I_i, i = 1, \ldots, \rho$is the partition of set $\Gamma = \{f_{s1}, \ldots f_{s\gamma}\}$ in to ρ disjoint set where $I_1 \cup \ldots \cup I_\rho = \Gamma$ and $I_i \cap I_j = \varphi$ for $1 \le i \ne j \le \rho$ and I_i can be an empty set.

Proposition 3 [23]: For two integers $\rho = 2$ as $\{F_1, . F_2\}$ with moduli $\Gamma = \{f_{s1}, \ldots f_{s\gamma}\}$ the largest dynamic range for unambiguous reconstruction from remainders is $F_{max} = \min_{I_1, I_2}\{lcm(I_1) + lcm(I_2)\} = \min_{I_1, I_2}\{\prod_{f_{sr} \in I_1} f_{sr} + \prod_{f_{sr} \in I_2} f_{sr}\}$ where $I_1 \cup I_2 = \Gamma$.

The largest dynamic range for single integer ($\rho = 1$) is lcm of all moduli i.e. $F_{max} = lcm(f_{s_1}, \ldots, f_{s_\gamma})$. It can be inferred from Proposition 1-3 that the dynamic range for multiple integers, in general, is less than lcm of all modules.

Thus, some works [22], [28] tried to achieve maximal possible range similar to single integer i.e lcm of all moduli.

To do this, they used some conditions on the multiple integers (input frequencies) or /and moduli (under-sampling frequencies) that was reviewed briefly at the following.

Proposition 4: If $F_\rho - F_1 < f_{s1}/2$ then $F_{max} = lcm(f_{s_1},\dots,f_{s_\gamma}) = \prod_{i=1}^{\gamma} f_{si}$.

Note that this paper achieves lcm of all moduli while the condition $F_\rho - F_1 < f_{s1}/2$ is admitted.

Proposition 5 [22]: If $F_\rho - F_1 < f_{s1}$, $GCD(\rho, f_{si}) = 1$; $1 \le i \le \gamma - 1$ and $\rho^2 - \rho(f_{s1} + l) + (l-1)f_{s\rho} > 0$ for $2 \le l \le \rho$ then $F_{max} = lcm(f_{s_1},\dots,f_{s_\gamma}) = \prod_{i=1}^{\gamma} f_{si}$ where $f_{s_1} <\dots< f_{s_\gamma}$ and GCD is the greatest common division.

Note that in this case, the difference between two disjoint integers (input frequencies) should be less than the minimum sampling frequency f_{s1} (i.e. $F_\rho - F_1 < f_{s1}$) while for Proposition 4, it is $f_{s1}/2$.

A multiple frequencies determination for narrow bandwidth signals when the maximum difference between input frequencies (multiple integers) are less than the maximum sampling frequency (moduli) i.e. $F_\rho - F_1 < f_{s\gamma}$ was also proposed in [21].

Proposed Approaches

Lemma 1 [23]: If two input frequencies sets $X = \{F_1,\dots,F_\rho\}$ and $Y = \{F_1',\dots,F_\rho'\}$ have the same remainder sets, i.e. $S_r(X) = S_r(Y)$, then the minimum of these integers would be zero, i.e. $min\{X \cup Y\} = 0$, and the maximum of these integers would be $max\{X \cup Y\} = F_{max}$ that F_{max} is a large dynamic range.

The order of remainders of each modulus is not known from output of DFT [19].

In other words if the ordered remainders set of multiple integers for r^{th} modulus be $S_r(F_1,\dots,F_\rho) = \overset{\rho}{\underset{l=1}{\cup}} \{f_{u(l,r)}\}, r = 1,\dots,\gamma$ then received the remainders set from output of DFT is $S_r'(F_1',\dots,F_\rho') = \overset{\rho}{\underset{l=1}{\cup}} \{t_{u(\delta_r(l),r)}\}, r = 1,\dots,\gamma$ where δ_r is an arbitrarily chosen onto mapping from index set $T = \{1,\dots,\rho\}$ to the indices of elements in $S_r(F_1,\dots,F_\rho)$. Note that when $\delta_r(l) = i$ we have $t_{u(\delta_r(l),r)} = t_{u(i,r)} = f_{u(i,r)}$.

It means l^{th} remainder of modulus r in ordered set of DFT, i.e. S_r', corresponding to the i^{th} integer, i.e. F_i, (See Table. 1).

Table 1: Assigning the remainders from remainder set of each modulus to integers

Integers	Mod f_{s1}	Mod f_{s2}	…	Mod $f_{s\gamma}$
F_1'	$t_{u(\delta_1(1),1)}$	$t_{u(\delta_2(1),2)}$	…	$t_{u(\delta_\gamma(1),\gamma)}$
F_2'	$t_{u(\delta_1(2),1)}$	$t_{u(\delta_2(2),2)}$	…	$t_{u(\delta_\gamma(2),\gamma)}$
…	…	…	…	…
F_ρ'	$t_{u(\delta_1(\rho),1)}$	$t_{u(\delta_2(\rho),2)}$	…	$t_{u(\delta_\gamma(\rho),\gamma)}$

We try to find integers F_l' based on its remainders $t_{u(\delta_r(l),r)}, r = 1,\dots,\gamma$ (see Table. 1). The relationship between an integer F_i and its remainders is as follows:

$$f_{u(i,r)} = F_i \bmod f_{sr} = F_i - k_{i,r} f_{sr} \tag{5}$$

where $k_{i,r} \in \{0,1,\dots,\lfloor F_{max} \;/f_{sr}\rfloor\}$.

The relationship between moduli f_{sr} ; $r = 1,\dots,p$ and $f_{u(l,r)}'$ as the remainder of F_l' is as follows:

$$f_{u(l,r)}' = F_l' \bmod f_{sr} = F_l' - k_{l,r}' f_{sr} \tag{6}$$

Proposition 6: Assume two frequencies sets $X = \{F_1,\dots,F_\rho\}$ and $Y = \{F_1',\dots,F_\rho'\}$ have the same under-sampled (remainder) sets i.e. $S_r(X) = S_r(Y)$ with sampling frequencies (moduli) $f_{sr}, r = 1,\dots,\gamma$. Now assume from all γ remainders for each $F_i \in X$ and $F_l' \in Y$ there are $\alpha_{(l,i)}$; $l = 1,\dots,\rho$; $i = 1,\dots,\rho$ common remainders (same remainders) with moduli $f_{sr_h^{(l,i)}}$; $h = 1,\dots,\alpha_{(l,i)}$ between $F_i \in X$ and $F_l' \in Y$. Then the difference value between $F_i \in X$ and $F_l' \in Y$ is as follows:

$$F_l' - F_i = k_{l,i}\, lcm\Big(\overset{\alpha_{(l,i)}}{\underset{h=1}{\cup}} f_{sr_h^{(l,i)}}\Big);$$
$$k_{l,i} \in \left\{0, \pm 1,\dots,\pm\left\lfloor F_{max} \Big/ lcm\big(f_{sr_1^{(l,i)}},\dots,f_{sr_{\alpha_{(l,i)}}^{(l,i)}}\big)\right\rfloor\right\} \tag{7}$$

Proof of Proposition 6: There are $\alpha_{(l,i)}$ same remainders between F_l' and F_i thus difference between these $\alpha_{(l,i)}$ remainders should be zero i.e. $f_{u(l,r_h^{(l,i)})}' - f_{u(i,r_h^{(l,i)})} = 0$; $h = 1,\dots,\alpha_{(l,i)}$. By considering (5) and (6) we have $f_{u(i,r_h^{(l,i)})}' - f_{u(i,r_h^{(l,i)})} = F_l' - F_i - k_{l,r_h^{li}}'' f_{sr_h^{(l,i)}} = 0$ where $k_{l,r}'' \in \{0, \pm 1,\dots,\pm\lfloor F_{max} \;/f_{sr}\rfloor\}$. So, it is possible to have following relationships:

$$F_l^{'} - F_i = k^{''}_{l,r_1^{(l,i)}} f_{sr_1^{(l,i)}} = \ldots = k^{''}_{l,r_h^{(l,i)}} f_{sr_h^{(l,i)}} = \ldots = k^{''}_{l,r_{\alpha(l,i)}^{(l,i)}} f_{sr_{\alpha(l,i)}^{(l,i)}} = \Lambda \quad (8)$$

From (8) it is obvious that Λ should be multiple of α_{li} moduli frequencies, i.e. $\Lambda / f_{sr_h^{(l,i)}} = k^{''}_{l,r_h^{(l,i)}}\,; h = 1,\ldots,\alpha_{(l,i)}$. Therefore, the smallest value that dividable to all moduli frequencies $f_{sr_1^{(l,i)}},\ldots,f_{sr_h^{(l,i)}},\ldots,f_{sr_{\alpha(l,i)}^{(l,i)}}$ is the least common multiple (lcm) of them i.e. $lcm(f_{sr_1^{(l,i)}},\ldots,f_{sr_h^{(l,i)}},\ldots,f_{sr_{\alpha(l,i)}^{(l,i)}})$. Thus, Λ is multiple of lcm of α_{li} moduli frequencies i.e. $\Lambda = k_{l,i} lcm(f_{sr_1^{(l,i)}},\ldots,f_{sr_h^{(l,i)}},\ldots,f_{sr_{\alpha(l,i)}^{(l,i)}})$. From Lemma 1 it is clear that $F_l^{i}, F_i \in [0, F_{max}\ ]$ thus $\Lambda = F_l^{'} - F_i = k_{l,i} lcm(\bigcup_{h=1}^{\alpha(l,i)} f_{sr_h^{(l,i)}})\,; k_{l,i} \in \{0, \pm 1, \ldots, \pm \left\lfloor F_{max}\ / lcm(f_{sr_1^{(l,i)}},\ldots,f_{sr_{\alpha(l,i)}^{(l,i)}})\right\rfloor\}$.

Proposition 7: Assume a set of ρ frequencies (integers) as$X = \{F_1,\ldots,F_\rho\}$ from under-sampled frequencies (remainders) with sampling frequencies (moduli) $f_{sr}, r = 1,\ldots,\gamma$ can be reconstructed unambiguously when $max(X) < F_{max}$ where F_{max} is called the largest dynamic range.

The largest dynamic range for ρ integers from remainders (frequencies) with moduli (sampling frequencies) $f_{sr}, r = 1,\ldots,\gamma$ can be obtained as follows:

$$F_{max} = max(\{F_1,\ldots,F_\rho\})| \sum_{l=1}^{\rho}\sum_{i=1}^{\rho}\left|F_l^{'} - F_i - k_{l,i} lcm(\bigcup_{h=1}^{\alpha(l,i)} f_{sr_h^{(l,i)}})\right| = 0 \;;\ \bigcup_{i=1}^{\rho}[\bigcup_{h=1}^{\alpha(l,i)} f_{sr_h^{(l,i)}}] = \Gamma, l = 1,\ldots,\rho \quad (9)$$

where $Y = \{F_1^{'},\ldots,F_\rho^{'}\}$ have the same remainders sets as X with moduli $f_{sr}, r = 1,\ldots,\gamma$, $Y \neq X$ and based on proposition 6 for each $F_i \in X$ and $F_l^{'} \in Y$ there are $\alpha_{(l,i)}\,;\ l = 1,\ldots,\rho\,;\ i = 1,\ldots,\rho$ common remainders (same remainders) with moduli $f_{sr_h^{(l,i)}}\,; h = 1,\ldots,\alpha_{(l,i)}$ between $F_i \in X$ and $F_l^{'} \in Y$. Since, X and Y have the same reminders.

Thus, common remainder between $F_i \in X$ and all $F_l^{'}s \in Y$ and compartment sampling frequencies $f_{sr_h^{(l,i)}}\,; h = 1,\ldots,\alpha_{(l,i)}$ should be Γ i.e. $\bigcup_{l=1}^{\rho}[\bigcup_{h=1}^{\alpha(l,i)} f_{sr_h^{(l,i)}}] = \Gamma$. Similar relation is existing between $F_l^{'}$'s$\in Y$ and all F_i's$\in X$ i.e. $\bigcup_{i=1}^{\rho}[\bigcup_{h=1}^{\alpha(l,i)} f_{sr_h^{(l,i)}}] = \Gamma$.

Proof of Proposition 7: Consider two different sets $X = \{F_1,\ldots,F_\rho\}$ and $Y = \{F_1^{'},\ldots,F_\rho^{'}\}$ have the same remainder sets with moduli $f_{sr}, r = 1,\ldots,\gamma$ where $max(X) \leq F_{max}$ and $max(Y) \leq F_{max}$. The $\alpha(l,i)$ is the number of common remainders between $F_l^{'}$ and F_i and $k_{l,i} \in \{0, \pm 1, \ldots, \pm \left\lfloor F_{max}\ / lcm(f_{sr_1^{(l,i)}},\ldots,f_{sr_{\alpha(l,i)}^{(l,i)}})\right\rfloor\}$ and Γ is the set of all moduli $\Gamma = \bigcup_{r=1}^{\gamma} f_{sr}$. Thus each $F_l^{'}$ has $\alpha(l,i)$ common remainder sets with F_i that $\sum_{i=1}^{\rho}\alpha(l,i) = \gamma$ or $\bigcup_{i=1}^{\rho}[\bigcup_{h=1}^{\alpha(l,i)} f_{sr_h^{(l,i)}}] = \Gamma$. Based on Proposition 6 the difference value is equal to (7). Thus, we can say $F_l^{'} - F_i - k_{l,i} lcm(\bigcup_{h=1}^{\alpha(l,i)} f_{sr_h^{(l,i)}}) = 0$. This relation must be fulfilled for a $F_l^{'}$ and all F_i's $i = 1,\ldots,\rho$ i.e. $\sum_{i=1}^{\rho}\left|F_l^{'} - F_i - k_{l,i} lcm(\bigcup_{h=1}^{\alpha(l,i)} f_{sr_h^{(l,i)}})\right| = 0$. Furthermore, this relationship should be satisfied for all $F_l^{'}$s $l = 1,\ldots,\rho$i.e. $\sum_{l=1}^{\rho}\sum_{i=1}^{\rho}\left|F_l^{'} - F_i - k_{l,i} lcm(\bigcup_{h=1}^{\alpha(l,i)} f_{sr_h^{(l,i)}})\right| = 0$.

In the following the proposed procedure is introduced to obtain the largest dynamic range F_{max} from (9). Note that when all under-sampling frequencies multiplied by constant c (increased c times) the lcm of under-sampling frequencies are also multiplied by c . Then, the maximum possible frequencies that satisfied (9) i.e. F_{max} will also be multiplied by c .

Procedure 1: The procedure for determination of the largest dynamic range can be summarized as follows:

Step 0*:* Initialize the largest dynamic range as $F_{max} = F_{max}^{Ini}$ in which F_{max}^{Ini} is greater than (e.g. ten times of) conventional dynamic range mentioned in proposition 2 i.e. $F_{max}^{Ini} \gg \min_{I_1\cup\ldots\cup I_\rho=\Gamma} max\{\prod_{f_{si}\in I_1} f_{si},\ldots,\prod_{f_{si}\in I_\rho} f_{si}\}$.

Step 1*:* Categorize moduli of $F_l^{'}$ (i.e. $\Gamma = \{f_{s1},\ldots f_{s\gamma}\}$) to ρ disjoint subsets as $a(l,i) = \bigcup_{h=1}^{\alpha(l,i)} f_{sr_h^{(l,i)}}, i = 1,\ldots,\rho$. The $\alpha(l,i)$ is a set of common moduli between $F_l^{'}$ and F_i and $a(l,i)$ is common moduli between $F_l^{'}$s and all F_i's. Since $a(l,i)$ s is obtained by categorizing γ moduli of $F_l^{'}$, it is possible to write $\bigcup_{i=1}^{\rho} a_{(l,i)} = \Gamma, l = 1,\ldots,\rho$.

Step 2*:* Common compartment modules between $F_l^{'}$'s and F_i's can be considered as a matrix:

$$\begin{array}{c} \\ F_1^{'} \\ \vdots \\ F_l^{'} \\ \vdots \\ F_\rho^{'} \end{array} \overset{\begin{array}{ccccc} F_1 & \cdots & F_i & \cdots & F_\rho \end{array}}{\begin{bmatrix} a_{(1,1)} & \cdots & a_{(1,i)} & \cdots & a_{(1,\rho)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(l,1)} & \cdots & a_{(l,i)} & \cdots & a_{(l,\rho)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(p,1)} & \cdots & a_{(p,i)} & \cdots & a_{(p,\rho)} \end{bmatrix}} \quad (10)$$

Based on Step 1 of the procedure each row is related to all $F_l^{'}$'s moduli. Thus, each row is chosen such that $\overset{i=1}{\underset{\rho}{\cup}} a_{(l,i)} = \Gamma, l = 1,\dots,\rho$. Each column related to all F_i's moduli. Thus, each column should check to be sure that $\overset{l=1}{\underset{\rho}{\cup}} a_{(l,i)} = \Gamma$, $i = 1,\dots,\rho$. If this condition is met, go to Step 3; otherwise, return to Step 1 and produce other possible moduli from Γ.

Step 3*:* Based on (9) and representation $a_{(l,i)}$in (10) following relationship can be written

$$F_l^{'} - F_i = k_{(l,i)} lcm(a(l,i)), \quad l = 1,\dots,\rho and i = 1,\dots,\rho \quad (11)$$

According to lemma 1, the F_1 should be zero. By considering $i = 1$ in (11) and $F_1 = 0$ we have the following relation:

$$F_l^{'} = k_{(l,1)} lcm(a(l,1)), l = 1,\dots,\rho \quad (12)$$

Now, by substituting (12) in (11), the F_i's for $i > 2$ can be obtained as below:

$$F_i = k_{(l,1)} lcm(a(l,1)) - k_{(l,i)} lcm(a(l,i)), \quad l = 1,\dots,\rho and i = 2,\dots,\rho \quad (13)$$

that $k_{(l,i)} \in \{0, \pm 1, \dots, \pm \lfloor F_{max} / lcm(a(l,i)) \rfloor\}$, $a(l,i) = \overset{h=1}{\underset{\alpha_{(l,i)}}{\cup}} f_{sr_h^{(l,i)}}, i = 1,\dots,\rho$ and also it is assumed that $F_{max} = F_{max}^{ini}$.

Step 4*:* Based on (11), each $F_l^{'}$ with each F_i's that has common moduli should meet $F_l^{'} = k_{(l,i)} lcm(a(l,i)) + F_i$, similar relations should be met for each F_i. When two sets $X = \{F_1,\dots,F_\rho\}$ and $Y = \{F_1^{'},\dots,F_\rho^{'}\}$ are found so that satisfy conditions in (11), we can consider $max(X)$ as final F_{max} and finish the process.

Otherwise, choose a bigger F_{max}^{Ini} e.g. double of previous F_{max}^{Ini} and go to step 1. It is notable, Proposition 7 presents a relationship to find the largest dynamic range (F_{max}) numerically by procedure1 for any ρ that not presented in the previous studies. However, procedure 1 can be simplified for some cases include $\rho = 2$ and $\rho = 3$. By considering two integers, i.e. $\rho = 2$, we show in Corollary 1 that the close form relationship for the largest dynamic range of two integers in [23] is a special case of proposed proposition 7.

Corollary 1: The largest dynamic range (maximum possible range of frequency for unique detection) for proposed proposition 7 when $\rho = 2$ (two frequencies) is $F_{max} = \min_{I_1 \cup I_2 = \Gamma} \{lcm(I_1) + lcm(I_2)\}$.

Proof of Corollary 1: For this case, the condition in (9) can be written as $\sum_{l=1}^{2} \sum_{i=1}^{2} \left| F_l^{'} - F_i - k_{l,i}\, lcm(\overset{\alpha_{(l,i)}}{\underset{h=1}{\cup}} f_{sr_h^{(l,i)}}) \right| = 0$. Thus, there are the following relationships:

$$\begin{aligned} F_1^{'} - F_1 &= k_{1,1} lcm(\overset{h=1}{\underset{\alpha_{(1,1)}}{\cup}} f_{sr_h^{(1,1)}}), \\ F_1^{'} - F_2 &= k_{1,2} lcm(\overset{h=1}{\underset{\alpha_{(1,2)}}{\cup}} f_{sr_h^{(1,2)}}), \\ F_2^{'} - F_1 &= k_{2,1} lcm(\overset{h=1}{\underset{\alpha_{(2,1)}}{\cup}} f_{sr_h^{(2,1)}}), \\ F_2^{'} - F_2 &= k_{2,2} lcm(\overset{h=1}{\underset{\alpha_{(2,2)}}{\cup}} f_{sr_h^{(2,2)}}) \end{aligned} \quad (14)$$

Let us show common compartment modules between $F_l^{'}$'s and F_i's as a matrix:

$$\begin{array}{c} \\ F_1^{'} \\ F_2^{'} \end{array} \overset{\begin{array}{cc} F_1 & F_2 \end{array}}{\begin{bmatrix} a_{(1,1)} & a_{(1,2)} \\ a_{(2,1)} & a_{(2,2)} \end{bmatrix}} \quad (15)$$

where $a_{(l,i)}$ is the common disjoint moduli between $F_l^{'}$ and F_i. Let's consider community of all moduli as $\Gamma = \overset{r=1}{\underset{\gamma}{\cup}} f_{sr}$. Thus, community between subsets of $F_1^{'}$ i.e. $a_{(1,1)}$ and $a_{(1,2)}$ in a row of mentioned matrix should be Γ i.e. $a_{(1,1)} \cup a_{(1,2)} = \Gamma$ similar for $F_2^{'}$, F_1 and F_2 there are $a_{(1,1)} \cup a_{(1,2)} = a_{(2,1)} \cup a_{(2,2)} = a_{(1,1)} \cup a_{(2,1)} = a_{(1,2)} \cup a_{(2,2)} = \Gamma$. These conditions are satisfied when $a_{(1,1)} = a_{(2,2)}$ and $a_{(1,2)} = a_{(2,1)}$. In other words, if all modules Γ are divided to two disjoint subsets I_1 and I_2 where $I_1 \cup I_2 = \Gamma$ then the matrix in (15) can be rewritten as:

$$\begin{array}{c} \\ F_1^{'} \\ F_2^{'} \end{array} \overset{\begin{array}{cc} F_1 & F_2 \end{array}}{\begin{bmatrix} I_1 & I_2 \\ I_2 & I_1 \end{bmatrix}} \quad (16)$$

Now, according to (14) and (16) and based on Lemma 1 by considering $F_1 = 0$ it can be written:

$$\begin{aligned} F_1^{'} &= k_{1,1} lcm(I_1), \\ F_2^{'} &= k_{2,1} lcm(I_2), \\ F_1^{'} - F_2 &= k_{1,2} lcm(I_2), \\ F_2^{'} - F_2 &= k_{2,2} lcm(I_1) \end{aligned} \quad (17)$$

To satisfy both relations $F_2 = k_{1,1}lcm(I_1) - k_{1,2}lcm(I_2)$ and $F_2 = k_{2,1}lcm(I_2) - k_{2,2}lcm(I_1)$ in (14) it should satisfy $k_{1,1} = k_{2,1} = 1$ and $k_{1,2} = k_{2,2} = -1$. Thus, $F_2 = lcm(I_1) + lcm(I_2)$ and the set of integers would be $X = \{0, F_2\}$ and F_{max} is the minimum possible of F_2, i.e. $F_{max} = \min_{I_1 U I_2 = \Gamma} max(X) = \min_{I_1 U I_2 = \Gamma}\{lcm(I_1) + lcm(I_2)\}$.

This, shows proposition 3 is a special case of proposed proposition 7 when $\rho = 2$.

Corollary 2: The common moduli between F_l''s and F_i, i.e. $a_{(l,i)} = \bigcup_{h=1}^{\alpha_{(l,i)}} f_{sr_h^{(l,i)}}$, for $\rho = 3$ that admit the conditions in proposition 7 can be simplified as follows:

$$\begin{array}{c} \\ F_1' \\ F_2' \\ F_3' \end{array}\begin{array}{c} \begin{array}{ccc} F_1 & F_2 & F_3 \end{array} \\ \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & a_{(1,3)} \\ a_{(2,1)} & a_{(2,2)} & a_{(2,3)} \\ a_{(3,1)} & a_{(3,2)} & a_{(3,3)} \end{bmatrix} \end{array}$$
$$\begin{array}{c} \\ F_1' \\ F_2' \\ F_3' \end{array}\begin{array}{c} \begin{array}{ccc} F_1 & F_2 & F_3 \end{array} \\ \begin{bmatrix} I_1 \cup I_2 & I_3 \cup I_6 & I_4 \cup I_5 \\ I_3 \cup I_4 & I_2 \cup I_5 & I_1 \cup I_6 \\ I_5 \cup I_6 & I_1 \cup I_4 & I_2 \cup I_3 \end{bmatrix} \end{array} \tag{18}$$

where $I_i, i = 1, \ldots, 6$ are disjoint subsets and $\bigcup_{i=1}^{6} I_i = \Gamma = \bigcup_{r=1}^{\gamma} f_{sr}$.

Note that, corollary 2 substitute the steps 1 and 2 of procedure 1 for the calculation $a_{(l,i)}$'s when $\rho = 3$.

Proof of Corollary 2: By considering the condition in (9), i.e. $\sum_{l=1}^{3}\sum_{i=1}^{3}\left|F_l' - F_i - k_{l,i}lcm(\bigcup_{h=1}^{\alpha_{(l,i)}} f_{sr_h^{(l,i)}})\right| = 0$, it is possible to write the following relations:

$$\begin{aligned} F_1' - F_1 &= k_{1,1}lcm(\bigcup_{h=1}^{\alpha_{(1,1)}} f_{sr_h^{(1,1)}}), \\ F_1' - F_2 &= k_{1,2}lcm(\bigcup_{h=1}^{\alpha_{(1,2)}} f_{sr_h^{(1,2)}}), \\ F_1' - F_3 &= k_{1,3}lcm(\bigcup_{h=1}^{\alpha_{(1,3)}} f_{sr_h^{(1,3)}}), \\ F_2' - F_1 &= k_{2,1}lcm(\bigcup_{h=1}^{\alpha_{(2,1)}} f_{sr_h^{(2,1)}}), \\ F_2' - F_2 &= k_{2,2}lcm(\bigcup_{h=1}^{\alpha_{(2,2)}} f_{sr_h^{(2,2)}}), \\ F_2' - F_3 &= k_{2,3}lcm(\bigcup_{h=1}^{\alpha_{(2,3)}} f_{sr_h^{(2,3)}}), \\ F_3' - F_1 &= k_{3,1}lcm(\bigcup_{h=1}^{\alpha_{(3,1)}} f_{sr_h^{(3,1)}}), \\ F_3' - F_2 &= k_{3,2}lcm(\bigcup_{h=1}^{\alpha_{(3,2)}} f_{sr_h^{(3,2)}}), \\ F_3' - F_3 &= k_{3,3}lcm(\bigcup_{h=1}^{\alpha_{(3,3)}} f_{sr_h^{(3,3)}}) \end{aligned} \tag{19}$$

Similar to corollary 1, the common compartment modules between F_l''s and F_i's can be shown as below matrix:

$$\begin{array}{c} \\ F_1' \\ F_2' \\ F_3' \end{array}\begin{array}{c} \begin{array}{ccc} F_1 & F_2 & F_3 \end{array} \\ \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & a_{(1,3)} \\ a_{(2,1)} & a_{(2,2)} & a_{(2,3)} \\ a_{(3,1)} & a_{(3,2)} & a_{(3,3)} \end{bmatrix} \end{array} \tag{20}$$

The community of moduli in all rows and columns should be admitted $\Gamma = \bigcup_{r=1}^{\gamma} f_{sr}$, as:

$$\begin{aligned} a_{(1,1)} \cup a_{(1,2)} \cup a_{(1,3)} \\ &= a_{(3,1)} \cup a_{(3,2)} \cup a_{(3,3)} \\ &= a_{(3,1)} \cup a_{(3,2)} \cup a_{(3,3)} \\ &= a_{(1,1)} \cup a_{(2,1)} \cup a_{(3,1)} \\ &= a_{(1,2)} \cup a_{(2,2)} \cup a_{(3,2)} \\ &= a_{(1,3)} \cup a_{(2,3)} \cup a_{(3,3)} \\ &= \Gamma \end{aligned} \tag{21}$$

To satisfy (21), the common moduli in matrix (20) can be expressed as follows:

$$\begin{array}{c} \\ F_1' \\ F_2' \\ F_3' \end{array}\begin{array}{c} \begin{array}{ccc} F_1 & F_2 & F_3 \end{array} \\ \begin{bmatrix} I_1 \cup I_2 & I_3 \cup I_6 & I_4 \cup I_5 \\ I_3 \cup I_4 & I_2 \cup I_5 & I_1 \cup I_6 \\ I_5 \cup I_6 & I_1 \cup I_4 & I_2 \cup I_3 \end{bmatrix} \end{array} \tag{22}$$

where $I_i, i = 1, \ldots, 6$ are disjoint subsets $I_i \subset \Gamma$and $\bigcup_{i=1}^{6} I_i = \Gamma = \bigcup_{r=1}^{\gamma} f_{sr}$. Now, the value $a_{(l,i)}$ that satisfied (21) can be used in steps 3 and 4 of Procedure 1 to find F_{max} . In fact, steps 1 and 2 of procedure 1 can be replaced by corollary 2 for calculation $a_{(l,i)}$s when $\rho = 3$.

Procedure 2: The procedure of determination input frequencies from their under-sampled frequencies for complex waveform. There are some similarities between the procedure of determination of frequencies from under sampled frequencies of a sinusoidal complex waveform (i.e. $\sum_{l=1}^{q} A_l e^{i(2\pi F_l t)} + w(t)$) and the determination of frequencies of real sinusoidal waveform (i.e. $\sum_{i=1}^{q} A_i \cos(2\pi F_i t + \varphi_i) + w(t)$) in [2]. However, should consider the fact that the under-sampled frequencies of the real waveform and the complex waveform are different as follows [2]:

$$f_{u(k,j)} = \begin{cases} (-1)^{\bar{v}_k}(F_j - m_{kj} f_{sk}); \quad \bar{v}_k \in \{1,2\} \\ \qquad \text{Realwaveform} \\ F_j - m_{kj} f_{sk} \text{Complexwaveform} \end{cases} \tag{23}$$

Thus, as can see in Fig. 2 (b) the under-sampled frequency curve for complex waveforms is not continues and a few noises or changes in frequency (F_j) can cause big change in $f_{u(k,j)}$ and reduce $f_{u(k,j)}$from maximum to zero or vice versa.

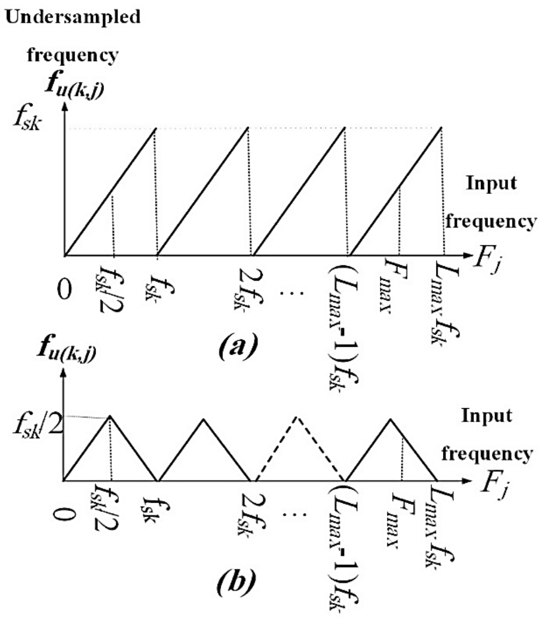

Fig. 2: Under-sampled frequency fu(k,j) as a function of j[th] analog input frequency $F_j \in [0, F_{max})$ after sampling with the k^{th} sampling frequency f_{sk} from (a) real signal waveform and (b) complex signal waveform.

Step 1: There are q frequencies that are sampled with p ADC's, thus there are $p \times q$ under-sampled frequencies as $\tilde{f}_{u(k,j)}$; $k = 1,\dots,p, j = 1,\dots,q$. However, the correspondences between q input frequencies and the q outputs under-sampled frequencies are unknown.

Thus, these $p \times q$ noisy under-sampled frequencies should be divided into q groups with p elements in each group as $\{S_1,\dots,S_j,\dots,S_q\} = \{\{\hat{f}_{u(i,1)}, i = 1,\dots,p\},\dots,\{\hat{f}_{u(i,j)}, i = 1,\dots,p\}, \quad \dots,\{\hat{f}_{u(i,q)}, i = 1,\dots,p\}\}$ in which the set $S_j = \{\hat{f}_{u(i,j)}, i = 1,\dots,p\}$; $j = 1,\dots,q$ denotes a noisy under-sampled frequencies set that corresponding to the j^{th} input frequency.

Step 2: Determines the distance $DIS_{\hat{f}_{u(i,j)}}$ for each set of $S_j = \{\hat{f}_{u(i,j)}, i = 1,\dots,p\}, j = 1,\dots,q$ as belows:

$$DIS_j = max(DIS_{\hat{f}_{u(1,j)}},\dots,DIS_{\hat{f}_{u(i,j)}}, \dots,DIS_{\hat{f}_{u(p,j)}}) \tag{24}$$

where the procedure for the computing of $DIS_{\hat{f}_{u(k,j)}}$ and $F_{est(\hat{f}_{u(k,j)})}$ for each set of S_j is described as follows:

Step 1: Calculate the frequencies $\hat{F}_k^t$s in the band $\hat{F}_k^t \in [0, F_{max})$ from $\hat{f}_{u(k,j)}$, when sampling frequency is f_{sk} as below:

$$\hat{F}_k^t = \hat{k}_k^t f_{sk} + \hat{f}_{u(k,j)}; \quad 0 \le \hat{k}_k^t f_{sk} < F_{max} \quad ; \quad \hat{k}_k^t = 0,1,\dots \tag{25}$$

Step 2: Determine under-sampled frequencies $\hat{f}_{ui}^t$; $i = 1,2,\dots,p, i \neq k$ related to $\hat{F}_k^t$'s when are sampled with sampling frequencies other than sampling frequency in step 1 i.e. f_{si}; $i = 1,2,\dots,p, i \neq k$.

$$\hat{F}_k^t = \hat{k}_i^t f_{si} + \hat{f}_{ui}^t \tag{26}$$

Step 3: Substitute $\hat{f}_{ui}^t$; $i = 1,2,\dots,p, i \neq k$ with their noisy under-sampled $\hat{f}_{u(i,j)}$; $i = 1,2,\dots,p, i \neq k$ in (26). Then calculate the following relationship:

$$\begin{aligned} \tilde{F}_i^t &= \left\{ \breve{F}_l^t \,\middle|\, \underset{l\in\{1,2,3\}}{\text{minimize}} \left|\breve{F}_l^t - \hat{F}_k^t\right| \right\}, \\ \breve{F}_1^t &= \hat{k}_i^t f_{si} + \hat{f}_{u(i,j)}, \\ \breve{F}_2^t &= (\hat{k}_i^t + 1) f_{si} + \hat{f}_{u(i,j)} \\ \breve{F}_3^t &= (\hat{k}_i^t - 1) f_{si} + \hat{f}_{u(i,j)} \end{aligned} \tag{27}$$

Note that unlike the under-sampled frequencies of the real waveform in the complex waveform small changes caused by noise can make a big change in the under-sampled frequencies as can be seen in Fig.2 (b). Thus, to substitute $\hat{f}_{ui}^t$s by $\hat{f}_{u(i,j)}$s should consider the $\hat{k}_i^t f_{si}$, $(\hat{k}_i^t + 1) f_{si}$ and $(\hat{k}_i^t - 1) f_{si}$.

Step 4: Find the $\hat{k}_i^t$s that minimize the following relationship and name them as $\hat{k}_i^{t^*}$:

$$\hat{k}_1^{t^*},\dots,\hat{k}_i^{t^*},\dots,\hat{k}_p^{t^*} = \{\hat{k}_1^t,\dots,\hat{k}_i^t,\dots,\hat{k}_p^t; \underset{\hat{k}_1^t,\dots,\hat{k}_i^t,\dots,\hat{k}_p^t}{min} max\{ |\tilde{F}_1^t - \tilde{F}_2^t|,\dots, |\tilde{F}_1^t - \tilde{F}_p^t|,\dots,|\tilde{F}_i^t - \tilde{F}_{i+1}^t|,\dots, |\tilde{F}_i^t - \tilde{F}_p^t|,\dots,|\tilde{F}_{p-1}^t - \tilde{F}_p^t|\}\}; \tag{28}$$

Definition 1: The maximum distance (DIS) between the frequencies $\tilde{F}_i^t$ in (27) related to $\hat{f}_{u(k,j)}$; $k = 1,\ldots,p$ is called $DIS_{\hat{f}_{u(k,j)}}$; $\hat{f}_{u(k,j)} \in S_j$ and defined as follows:

$$DIS_{\hat{f}_{u(k,j)}} \overset{\Delta}{=} max\{|\tilde{F}_1^t - \tilde{F}_2^t|,\ldots,|\tilde{F}_1^t - \tilde{F}_p^t|,\ldots, |\tilde{F}_i^t - \tilde{F}_{i+1}^t|,\ldots,|\tilde{F}_i^t - \tilde{F}_p^t|,\ldots, |\tilde{F}_{p-1}^t - \tilde{F}_p^t|\}; \quad \hat{k}_1^t = \hat{k}_1^{t^*},\ldots,\hat{k}_p^t = \hat{k}_p^{t^*}, i = 1,2,\ldots,p \tag{29}$$

Step 5: The estimated input frequency is obtained by mean of the frequencies ($\tilde{F}_i^t$s) that minimize (28) as below:

$$F_{est(\hat{f}_{u(k,j)})} = \sum_{i=1}^{p} \tilde{F}_i^t / p; \quad \hat{k}_1^t = \hat{k}_1^{t^*},\ldots,\hat{k}_p^t = \hat{k}_p^{t^*}, i = 1,2,\ldots,p, \tag{30}$$

Step 3. Obtain the possible input frequencies for state of each set in S_j; $j = 1,\ldots,q$ as $F_{state(n)} = \{F_{est(\hat{f}^*_{u(i_1,1)})},\ldots,F_{est(\hat{f}^*_{u(i_j,j)})},\ldots,F_{est(\hat{f}^*_{u(i_q,q)})}\}$ where $F_{est(\hat{f}_{u(i_j,j)})}$ was calculated in (30) and $\hat{f}^*_{u(i_j,j)}$ is ith under-sampled frequency of jth input frequency which minimizes the defined distance in (29) i.e. $DIS_{\hat{f}^*_{u(i_j,j)}} = \min_{\hat{f}_{u(i,j)}}(DIS_{\hat{f}_{u(1,j)}},\ldots,DIS_{\hat{f}_{u(p,j)}})$.

Step 4. Repeat steps 1 to 3 for all different states of dividing $p \times q$ under-sampled frequencies in to q groups with p elements in each group. In other words, steps 1 to 3 should be carried out for $n = 1,\ldots,(p!)^{q-1}$ different states.

Find the state that has the minimum value of $DIS_{state(n)}$ as below:

$$DIS_{state(n^*)} = \min_n(DIS_{state(1)},\ldots, DIS_{state(n)},\ldots) \tag{31}$$

The correct input analog frequencies are obtained based on n^* in (31) as $F_{state(n^*)}$.

Proposition 8: The maximum tolerable noise that multiple input frequencies $F_i \in [0, F_{max}]; i = 1,\ldots,\rho$ from noisy under-sampled frequencies $\tilde{f}_{u(r,i)} = f_{u(r,i)} + \varepsilon_{(r,i)}, r = 1,\ldots,\gamma; i = 1,\ldots,\rho$, with sampling frequencies $f_{sr}, r = 1,\ldots,\gamma$ is $\varepsilon_{max(Tolerable)} = \chi_{min}/4$. It is notable F_{max} is the largest possible range that obtained in proposition 7 and not a large range as the previous works.

The noise of each under-sampled frequency ($\varepsilon_{(r,i)}$) and the maximum noise of all under-sampled frequencies (ε_{max}) should be less than the maximum tolerable noise as $\varepsilon_{(r,i)} \leq \varepsilon_{max} \leq \varepsilon_{max(Tolerable)} = \chi_{min}/4$. Where, in this proposition, the frequency $f_{u(r,i)}$ is a noiseless under sampled frequency, $\varepsilon_{(r,i)}$ is an additive noise, ε_{max} is the maximum value of all $\varepsilon_{(r,i)}$'s and,

$$\chi_{min} = 4\varepsilon_{max(Tolerable)} = \min_{\substack{\{f_{si_1},f_{si_2}\}\subset\{f_{s1},f_{s2},\ldots,f_{s\gamma}\};\\ k'_{i_1},k'_{i_2},\\ |k'_{i_1}f_{si_1}-k'_{i_2}f_{si_2}|\neq 0}} |k'_{i_1}f_{si_1} - k'_{i_2}f_{si_2}|; \quad 1 \leq i_1 < i_2 \leq \gamma \tag{32}$$

Note that, the $k'_{i_t}, t \in \{1,2\}$ are some integers in (32) can be selected as $k'_{i_t} \in \{0, \pm 1,\ldots,\pm k_{i_t}^{max}\}$; $(k_{i_t}^{max} - 1)f_{si_t} < F_{max} < k_{i_t}^{max} f_{si_t}$ or $k_{i_t}^{max} = \lceil F_{max}/f_{si_t} \rceil$ where F_{max} is defined as (4).

Proof of Proposition 8: Consider a frequency F under-sampled with $r = 1,\ldots,p$ sampling frequency as follows:

$$F = \bar{k}_1 f_{s1} + f_{u(1,j)} = \ldots = \bar{k}_r f_{sr} + f_{u(r,j)} = \ldots = \bar{k}_p f_{sp} + f_{u(p,j)} \tag{33}$$

where $\bar{k}_r$ is correct integer that relates noiseless under-sampled frequency $f_{u(r,j)}$ to F.

Based on (29) the $DIS_{\hat{f}_{u(k,j)}}$ is the distance between p estimations of F_j from p available under-sampled frequencies i.e. $S_j = \{\hat{f}_{u(r,j)}, r = 1,\ldots,p\}$.

Consider the distance $|\tilde{F}_i^t - \tilde{F}_l^t|$ in $DIS_{\hat{f}_{u(k,j)}}$ in (29) as $D_{il} = |\tilde{F}_i^t - \tilde{F}_l^t|$.We prove that D_{il}, for not incorrect chosen of is greater than $\bar{D}_{il} = |\bar{\tilde{F}}_i^t - \bar{\tilde{F}}_l^t|$ where $\bar{\tilde{F}}_i^t$ and $\bar{\tilde{F}}_l^t$ are the correct estimated frequency of $\tilde{F}_i^t$ and $\tilde{F}_l^t$, respectively. In other words $\bar{\tilde{F}}_i^t$; $i = 1,\ldots,p$ are $\tilde{F}_i^t = k_i f_{si} + \hat{f}_{u(i,j)}$ that $\hat{f}_{u(i,j)}$ is noisy under-sampled frequencies and $k_i$are equal to the correct one i.e. $\bar{k}_i$ in (33) or $\bar{\tilde{F}}_i^t = \bar{k}_i f_{si} + \hat{f}_{u(i,j)}$.

We have $\hat{f}_{u(i,j)} = f_{u(i,j)} + \varepsilon_{(i,j)}$, $\hat{f}_{u(l,j)} = f_{u(l,j)} + \varepsilon_{(l,j)}$, $\tilde{F}_i^t = k_i f_{sr} + \hat{f}_{u(r,j)} = k_i f_{si} + f_{u(i,j)} + \varepsilon_{(i,j)}$, $\tilde{F}_l^t = k_l f_{sl} + f_{u(l,j)} + \varepsilon_{(l,j)}$and substituting $f_{u(i,j)} - f_{u(l,j)} = \bar{k}_j f_{sj} - \bar{k}_i f_{si}$ from (33) have the following equation:

$$D_{il} = |\tilde{F}_i^t - \tilde{F}_l^t| = \left|(k_i - \bar{k}_i)f_{si} - (k_l - \bar{k}_l)f_{sl} + \varepsilon_{(i,j)} - \varepsilon_{(l,j)}\right| = \left|k'_i f_{si} - k'_l f_{sl} + \varepsilon_{(i,j)} - \varepsilon_{(l,j)}\right| \tag{34}$$

Now, there are two states. For the correct estimation we have $k_i = \bar{k}_i$, $k_l = \bar{k}_l$ thus $k'_i = 0$ and $k'_l = 0$ and D_{il} in (34) can be rewritten:

$$D_{il} = \left|\varepsilon_{(i,j)} - \varepsilon_{(l,j)}\right| \leq 2\varepsilon_{max} \quad ; \quad for\ the\ correct\ estimation \tag{35}$$

For incorrect estimation $k'_i \neq 0$ and $k'_l \neq 0$. Thus, for the incorrect estimation can write:

$$D_{il} = \left|k'_i f_{si} - k'_l f_{sl} + \varepsilon_{(i,j)} - \varepsilon_{(l,j)}\right| \geq \left|k'_i f_{si} - k'_l f_{sl}\right| - \left|\varepsilon_{(i,j)} - \varepsilon_{(l,j)}\right| \geq \left|k'_i f_{si} - k'_l f_{sl}\right| - 2\varepsilon_{max} \tag{36}$$

Based on (32) we have $\left|k_r^{'} f_{sr} - k_s^{'} f_{ss}\right| > 4\varepsilon_{max}$. Thus, for the incorrect estimation can write:

$$D_{rs} \geq 4\varepsilon_{max} \quad - 2\varepsilon_{max} \quad = 2\varepsilon_{max} \quad ; \quad for\ the\ incorrect\ estimation \tag{37}$$

When one of F_i's is estimated incorrectly $D_{il} = |\tilde{F}_i^t - \tilde{F}_l^t| \geq 2\varepsilon_{max}$ in $DIS_{\hat{f}_{u(k,j)}}$ then $DIS_{\hat{f}_{u(k,j)}} \geq 2\varepsilon_{\max}$ or can write:

$$\begin{cases} DIS_{\hat{f}_{u(k,j)}} \leq 2\varepsilon_{\max} & \text{for the correct estimation} \\ DIS_{\hat{f}_{u(k,j)}} \geq 2\varepsilon_{\max} & \text{for the incorrect estimation} \end{cases} \tag{38}$$

It means by minimizing $DIS_{\hat{f}_{u(k,j)}}$ in (29) as (31) when noises are less that $\varepsilon_{max(Tolerable)}$ in (32) the frequencies can be determined uniquely.

Results and Discussion

A. The Maximum Possible Dynamic Range of Under-Sampling Frequency Detection

To demonstrate the proposed approach, consider the largest dynamic range for $\rho = 2$ input frequencies (integers) with $\gamma = 6$ sensors and sampling frequencies (moduli) $\Gamma = \overset{r=1}{\underset{\gamma}{\cup}} f_{sr} = \{3,5,7,11,13,17\}$Hz. According to Proposition 1, the dynamic range is $F_{max} = \min_{1 \leq r_1 \leq \ldots \leq r_3 \leq \gamma} lcm\{f_{s(r_1)}, \ldots, f_{s(r_3)}\} = lcm\{3,5,7\} = 105$. From Proposition 2 the dynamic range is $F_{max} = \min_{I_1 \cup \ldots \cup I_\rho = \Gamma} max\{\prod_{f_{si} \in I_1} f_{si}, \ldots, \prod_{f_{si} \in I_\rho} f_{si}\} = 516$ for $I_1 = \{3,11,17\}$ and $I_2 = \{5,7,13\}$. Based on (17) $F_1^{'} = k_{1,1} lcm(I_1)$ and $F_2^{'} = k_{2,1} lcm(I_2)$, and F_2 can be obtained from two formulas $F_2 = k_{1,1} lcm(I_1) - k_{1,2} lcm(I_2)$ and $F_2 = k_{2,1} lcm(I_2) - k_{2,2} lcm(I_1)$. Based on Procedure 1 the minimum possible value for F_2 that satisfies both formulas are obtained when $k_{1,1} = 1$, $k_{2,1} = 1$, $k_{2,1} = -1$, $k_{2,2} = -1$, $I_1 = \{3,11,17\}$, and $I_2 = \{5,7,13\}$. Thus, $F_1^{'} = 561$Hz, $F_2^{'} = 455$Hz and $F_2 = 1016$Hz. Two sets $X = \{F_1, F_2\} = \{0,1016\}$ and $Y = \{F_1^{'}, F_2^{'}\} = \{561,455\}$ have the same remainders and $F_{max} = max(X) = 1016$Hz. Similarly, based on Corollary 1 we have $F_{max} = \min_{I_1 U I_2 = \Gamma}\{lcm(I_1) + lcm(I_2)\}$ where $I_1 = \{3,11,17\}$, $I_2 = \{5,7,13\}$, $F_{max} = 1016$Hz. The dynamic range of two integers without conditions on them for the proposed approach and previous works has been shown in Table. 2.

As discussed previously Proposition 3 [23] which is just for two integers is a special case of proposed proposition 7 when $\rho = 2$.

Table. 2: The dynamic range for two frequencies (integers)

Approach	Dynamic range
Proposition 1 [29], [30]	105
Proposition 2 [19]	561
Proposition 3 [17], [23], available just for two integers	1016
Proposition 7 (proposed approach)	1016

Now, consider the largest dynamic range for $\rho = 3$ input frequencies (integers) for moduli $\Gamma = \{3,5,7,11,13,17\}$. Based on corollary 2 each six disjoint partitions of Γ i.e. $\overset{i=1}{\underset{6}{\cup}} I_i = \Gamma$ in matrix form in (18) will satisfy (21) or, equivalently, admit the conditions $\overset{i=1}{\underset{\rho}{\cup}} a_{(l,i)} = \overset{i=1}{\underset{\rho}{\cup}} [\overset{h=1}{\underset{\alpha(l,i)}{\cup}} f_{sr_h^{(l,i)}}] = \Gamma, l = 1, \ldots, \rho$; $\overset{l=1}{\underset{\rho}{\cup}} a_{(l,i)} = \overset{l=1}{\underset{\rho}{\cup}} [\overset{h=1}{\underset{\alpha(l,i)}{\cup}} f_{sr_h^{(l,i)}}] = \Gamma, i = 1, \ldots, \rho$ in proposition 7. Now, the obtained $a_{(l,i)}$'s by corollary 2 can be used in steps 3 and 4 of Procedure 1 to find F_{max} in the following. By considering initial largest dynamic range as $F_{max}^{Ini} = 1000$Hz in procedure 1, we have $I_1 = \{f_{s4}\}$, $I_2 = \{f_{s1}, f_{s3}\}$, $I_3 = \{f_{s5}\}$, $I_4 = \phi$, $I_5 = \{f_{s2}\}$, and $I_6 = \{f_{s6}\}$. Thus, based on (18) there is following relation:

$$\begin{array}{c c} & \begin{array}{ccc} F_1 & F_2 & F_3 \end{array} \\ \begin{array}{c} F_1^{'} \\ F_2^{'} \\ F_3^{'} \end{array} & \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & a_{(1,3)} \\ a_{(2,1)} & a_{(2,2)} & a_{(2,3)} \\ a_{(3,1)} & a_{(3,2)} & a_{(3,3)} \end{bmatrix} \end{array} \tag{39}$$

$$\begin{array}{c c} & \begin{array}{ccc} F_1 & F_2 & F_3 \end{array} \\ \begin{array}{c} F_1^{'} \\ F_2^{'} \\ F_3^{'} \end{array} & \begin{bmatrix} \{f_{s4}, f_{s1}, f_{s3}\} & \{f_{s5}, f_{s6}\} & \{f_{s5}, f_{s6}\} \\ \{f_{s5}\} & \{f_{s1}, f_{s3}, f_{s2}\} & \{f_{s4}, f_{s6}\} \\ \{f_{s2}, f_{s6}\} & \{f_{s4}\} & \{f_{s1}, f_{s3}, f_{s5}\} \end{bmatrix} \end{array}$$

The $k_{(i,j)}$'s that satisfy (11) and (19) are as:

$$\begin{array}{c c} & \begin{array}{ccc} F_1 & F_2 & F_3 \end{array} \\ \begin{array}{c} F_1^{'} \\ F_2^{'} \\ F_3^{'} \end{array} & \begin{bmatrix} k_{(1,1)} & k_{(1,2)} & k_{(1,3)} \\ k_{(2,1)} & k_{(2,2)} & k_{(2,3)} \\ k_{(3,1)} & k_{(3,2)} & k_{(3,3)} \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 1 & -131 \\ 25 & 3 & -3 \\ 4 & 30 & -2 \end{bmatrix} \tag{40}$$

Based on Step 3 of Procedure 1 the $F_1 = 0$ and $F_1^{'} - F_1 = k_{(1,1)} lcm(a(1,1)) = k_{(1,1)} lcm(\{f_{s4}, f_{s1}, f_{s3}\}) = 1 \times$

$lcm(\{11,3,7\}) = 231$ and $F_1^{'} = 231$. Other frequencies also is obtained based on Procedure 1. For the obtained $X = \{F_1, F_2, F_3\} = \{0,10,886\}$ and $Y = \{F_1^{'}, F_2^{'}, F_3^{'}\} = \{231,325,340\}$, the largest dynamic range is $F_{max} = max(X) = 886$Hz which for all $F_l^{'}$'s and F_i's should satisfies (11), i.e. $F_l^{'} - F_i = k_{(l,i)} lcm(a(l,i))$; $l = 1,\dots,\rho, i = 1,\dots,\rho$. As an example, for $F_2^{'} - F_3 = k_{(2,3)} lcm(a(2,3))$ we have $325 - 886 = -3 \times lcm(\{11,17\})$. It is notable that, the large dynamic range by previous studies based on proposition 2 is $F_{max} = \underset{I_1 \cup \dots \cup I_\rho = \Gamma}{min}\ max\{\prod_{f_{si} \in I_1} f_{si}, \dots, \prod_{f_{si} \in I_\rho} f_{si}\} = max\{lcm(\{3,17\}), lcm(\{5,13\}), lcm(\{7,11\})\} = 77$. The dynamic range of three integers without conditions on them for the proposed approach and previous works has been shown in Table. 3.

Table. 3: The dynamic range for three frequencies (integers)

Approach	Dynamic range
Proposition 1 [19], [30]	17
Proposition 2 [19]	77
Proposition 7 (proposed approach)	886

For previous works, the large dynamic range for unambiguous reconstruction of input frequencies is $F_{max} = 77$Hz while the larges dynamic range obtained by proposed approach is $F_{max} = 886$ Hz that is 11.5 times greater than the previous works.

Assume a digital instance frequency measurement (DIFM) equipped to ADCs with sampling rates $\Gamma = \bigcup_{r=1}^{\gamma} f_{sr} = \{3,5,7,11,13,17\} \times 10^7 = \{30,50,. 70,110,130,170\} \times MHz$what is the maximum possible range when 3 input frequencies come simultaneously? Before our work designer could claim designed DIFM guarantees reconstruction 3 simultaneous input frequencies uniquely until $77 \times 10^7 Hz = 770MHz$ now based on maximum upper bound obtained by proposed Proposition 7 can claim DIFM can reconstruct frequencies uniquely until$886 \times 10^7 Hz = 8.86GHz$. For the user of DIFM is also important to know for a higher range of frequency can guarantee to reconstruct frequencies.

B. The Under-Sampling Frequency Estimation for Noisy Waveform

This section, simulates the effect of noises on frequency estimations when sampling frequencies are very low.

The simulations are conducted for appropriate and non-appropriate under-sampling frequencies. For the first simulation, the maximum bound of frequencies is considered as a large bound obtained in the previous works for $\rho = 3$ input frequencies and low sampling frequencies $\Gamma = \{3,5,7,11,13,17\}$Hz.

A large bound as shown in Table. 3 for the previous works is 77 Hz. The maximum tolerable noise for this bound based on Theorem 2 of [2] for complex waveform (not real waveform) and for three input frequencies is 0.6.

In this work, we could find the maximum possible range for unique detection of multiple frequencies when sampling with very low sampling frequencies in Proposition 7 that for mentioned sampling frequencies (i.e. Γ) is obtained 886Hz in the previous section. Simulations have been done for 100000 random frequencies per each upper bound of noise for under-sampled frequencies. For the previous works a large obtained dynamic range that guarantee of unique detection was 77 Hz. Thus, random input frequencies are chosen in the range $[0,77)$ in Fig. 3. The newly obtained upper bound frequency for unique detection of input frequencies is 886Hz. Consequently, random input frequencies are chosen in the range $[0,886)$ in Fig. 4.

The procedure for detection frequencies was introduced in Procedure 2. The maximum tolerable noise for the proposed approach for three input frequencies and Γ under-sampling frequencies when $F_{max} = 886$ based on proposed Proposition 8 is $\varepsilon_{max(Tolerable)} = \frac{\chi_{min}}{4} = \frac{2}{4} = 0.25$.

Thus, for the proposed approach the maximum unique detectable frequencies and the maximum tolerable frequency noises are 886Hz and 0.25Hz against 77 Hz and 0.6 Hz for the previous works. For non-appropriate under-sampling frequencies like $\Gamma_{non-appropriate} = \{5,6,8,12,15,18\}$Hz that are greater than their counterpart sampling frequencies in Γ but the maximum tolerable noise for this set and $F_{max} = 886$ based on proposed Proposition 8 is $\varepsilon_{max(Tolerable)} = \frac{\chi_{min}}{4} = \frac{0}{4} = 0$. Thus, for non-appropriate low sampling frequencies even without the noise we cannot detect frequencies uniquely as shown in Fig. 5.

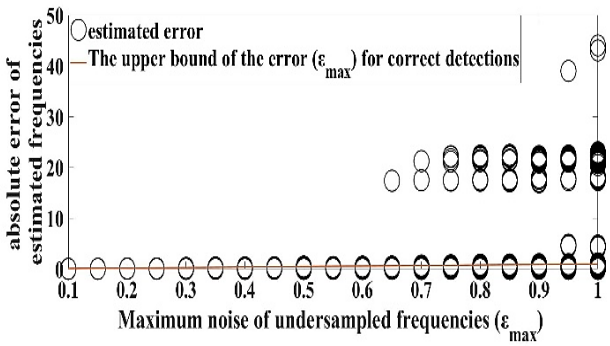

Fig. 3: Under-sampling frequency detection for noisy under-sampled frequencies of multiple input frequencies within range $[0,77)$ and appropriate sampling frequencies Γ.

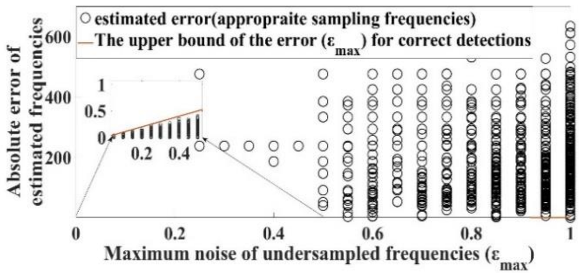

Fig. 4: Under-sampling frequency detection for noisy under-sampled frequencies of multiple input frequencies with range $[0,886)$ (more than 11 times greater range than previous studies) and appropriate sampling frequencies Γ.

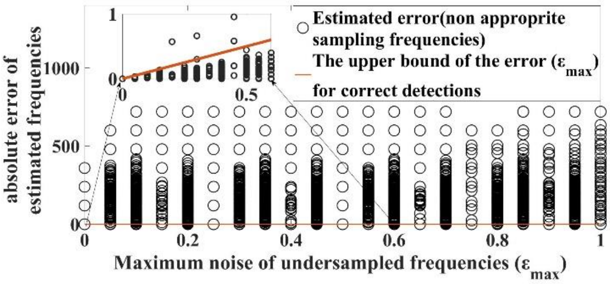

Fig. 5: Under-sampling frequency detection for noisy under-sampled frequencies for multiple input frequencies with range $[0,886)$ and non-appropriate sampling frequencies $\Gamma_{non-appropriate}$.

Conclusion

This study proposed propositions and a procedure to find the largest possible dynamic range for frequencies in a sinusoidal waveform with any number of frequencies for the unambiguous reconstruction of the frequencies of the waveform with very low sampling rates.

Furthermore, the proposed propositions were specified and simplified for waveforms with two and three frequencies and showed that the previous works for the maximum possible range for reconstruction frequencies of waveforms with two frequencies are a special case of our work.

It has been shown that for some cases the proposed approach could achieve 11.5 times greater dynamic range for the unambiguous reconstruction the frequencies of an under-sampled waveform with very low sampling rates.

A procedure for multiple frequencies detection from reminders (under-sampled frequencies) was proposed and maximum tolerable noises of under-sampled frequencies for unique detection were obtained. There are two main disadvantages when using of under-sampling approaches.

When error in under-sampled frequencies is more than tolerable noise the origin frequencies cannot be reconstructed uniquely.

It is also necessary to have a computation unite to reconstruct origin frequencies from under-sampled frequencies. However, using the under-sampling approaches is obligatory in some situations such as the sampling rates of ADCs are very less than the range of frequencies or because of energy consumption or price cannot use more ADCs to cover a high range of frequencies.

In this study, the maximum upper bound for any number of the input frequencies for complex waveform was investigated. In some applications, direct sampling from the real waveform is needed because of hardware limitations.

The relation between actual frequencies and under-sampled frequencies from under-sampled waveform different for complex sampling and real (direct) sampling. Finding the maximum upper bound for any number of input frequencies from directly under-sampled waveform (none complex waveform) is suggested for future work.

Author Contributions

Dr. Ali Maroosi. and Dr. Hossein Khaleghi Bizaki suggested the model and innovation of the problem. Simulation has been done by A. Maroosi. The first draft was written by A. Maroosi and reviewed by H. Khaleghi Bizaki.

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Conflict of Interest

The authors declare no potential conflict of interest regarding the publication of this work. In addition, the ethical issues including plagiarism, informed consent, misconduct, data fabrication and, or falsification, double publication and, or submission, and redundancy have been completely witnessed by the authors.

Abbreviations

ADC	Analog-to-Digital Converter
CRT	Chinese Remainder Theorem
DOA	Direction of Arrival
NOMA	non-orthogonal multiple access
UAV	unmanned aerial vehicle
lcm	least common multiple
DFT	discrete Fourier transform

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