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Research Paper

Central indices energy of special graphs

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Abstract. Given a graph G with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let d_i be the degree of the vertex v_i in G for $i = 1, 2, \cdots, n$. We introduce the sum of degrees and the product of degrees matrices of a graph. Furthermore, we consider the central indices matrix as an Arithmetic mean matrix, Geometric mean matrix, and Harmonic mean matrix. The spectral of these matrices has been computed. In this paper, we investigate the central indices energy of some classes of graphs and several results concerning its energy have been obtained.

Keywords: eigenvalue of a graph, energy, geometric mean energy, arithmetic mean energy, harmonic mean energy
Mathematics Subject Classification (2010): 05C50, 05C99.

1 Introduction

One of the most practical applications of graph theory to chemistry is in the creation of molecular descriptors. This kind of descriptor is based on invariants obtained from the representation of molecular structures like graphs. Most of these invariants are induced from the adjacency or distance matrices of the graphs. Significant classes of these descriptors are topological indices, which are simple numbers obtained by the mathematical methods of graphs associated with molecules. There are more than 120 topological indices but only at most a dozen have been applied. These can be classified by the structural properties of the

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graphs used for their calculation [5].

In this paper we are concerned with simple finite graphs, without directed, multiple, or weighted edges, and without self-loops. Let $A(G)$ be adjacency matrix of G and $\lambda_1, \lambda_2, \cdots, \lambda_n$ its eigenvalues. These are said to be the eigenvalues of the graph G and to form its spectrum. Graph energy is an invariant that is calculated from the eigenvalues of the adjacency matrix of the graph. In the mathematical literature, The concept of energy was formally put forward in 1978 [6]. Then, a dramatic change occurred, and graph energy started to attract the attention of a remarkably large number of mathematicians. The energy $E(G)$ of the graph G is defined as the sum of the absolute values of its eigenvalues.

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For more details on the mathematical aspects of the theory of graph energy see [1,7,16,17]. Nowadays, due to the importance of applications of topological indices in chemistry, many researchers have studied these indices [8,14]. Hence, one of the most well known is Wiener index [9,18] is based on distance of vertices in the graph, the Hosoya index [12,13] is calculated by counting non-incident edges in a graph, the energy and the Estrada index [10] are based on the spectrum of the graph, the Randic connectivity index [2,11] and the Zagreb group indices [3,15] are calculated using the degrees of vertices, geometric-arithmetic indices [4] is based on some properties of vertices of graph, etc.

Motivated by the mentioned papers, we study by focusing on the Central Indices energy of graphs. Plus, the energies of a number of well-known and much studied families of graphs are computed.

Let $G = (V, E)$ be a simple graph of order n with vertex set $V = V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = E(G)$. Let d_i be the degree of the vertex v_i in G for $i = 1, 2, \cdots, n$. The Sum of degrees matrix $SD = SD(G)$ is

$$SD_{ij} = \begin{cases} d_i + d_j & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

The Arithmetic mean matrix $AM = AM(G)$ is defined as

$$AM_{ij} = \begin{cases} \dfrac{d_i + d_j}{2} & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

The Product of degrees matrix $PD = PD(G)$ is

$$PD_{ij} = \begin{cases} d_i d_j & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

The Geometric mean matrix $GM = GM(G)$ is defined by

$$GM_{ij} = \begin{cases} \sqrt{d_i d_j} & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the Harmonic mean matrix $HM = HM(G)$ is defined by

$$HM_{ij} = \begin{cases} \frac{2d_i d_j}{d_i + d_j} & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

The above matrices are real symmetric, so we can order the eigenvalues of their matrices so that $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The Central Indices energy related to all above mention matrices of graph G, denoted by $CIE(G)$, is defined as

$$CIE(G) = \sum_{i=1}^{n} |\rho_i|.$$

2 Central indices energy of families of well-known graphs

In this section, we investigate the central indices energy of well-known of graphs. For $n \geq 3$, the Central indices energy of the Star graph S_n is as following table

Matrix name	Spectral	$E_{CI}(G)$
A	$\begin{pmatrix} \sqrt{n-1} & 0 & -\sqrt{n-1} \\ 1 & n-2 & 1 \end{pmatrix}$	$2\sqrt{n-1}$
SD	$\begin{pmatrix} n\sqrt{n-1} & 0 & -n\sqrt{n-1} \\ 1 & n-2 & 1 \end{pmatrix}$	$2n\sqrt{n-1}$
AM	$\begin{pmatrix} \frac{n\sqrt{n-1}}{2} & 0 & -\frac{n\sqrt{n-1}}{2} \\ 1 & n-2 & 1 \end{pmatrix}$	$n\sqrt{n-1}$
PD	$\begin{pmatrix} (n-1)\sqrt{n-1} & 0 & -(n-1)\sqrt{n-1} \\ 1 & n-2 & 1 \end{pmatrix}$	$2(n-1)\sqrt{n-1}$
GM	$\begin{pmatrix} n-1 & 0 & -(n-1) \\ 1 & n-2 & 1 \end{pmatrix}$	$2(n-1)$
HM	$\begin{pmatrix} \frac{2(n-1)\sqrt{n-1}}{n} & 0 & -\frac{2(n-1)\sqrt{n-1}}{n} \\ 1 & n-2 & 1 \end{pmatrix}$	κ

Where $\kappa = \frac{4(n-1)\sqrt{n-1}}{n}$.

Proof. We can see that the adjacency matrix of S_n as follow,

$$A(S_n) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Also, $SD = nA, AM = \frac{n}{2}A, PD = (n-1)A, GM = \sqrt{n-1}A, HM = \left(\frac{2(n-1)}{n}\right)A$. With simple computation, we can get the results inside the table. $\square$

We denote the complete graph on n vertices By K_n, and by $\bar{K_n}$ its complement, i.e., the graph consisting of n isolated vertices. If G is the complete graph K_n then,

Matrix name	Spectral	$E_{CI}(G)$
A	$\begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$	$2(n-1)$
SD	$\begin{pmatrix} 2(n-1)^2 & 2(n-1)^2 \\ 1 & n-1 \end{pmatrix}$	$4(n-1)^2$
AM	$\begin{pmatrix} (n-1)^2 & -(n-1) \\ 1 & n-1 \end{pmatrix}$	$2(n-1)^2$
PD	$\begin{pmatrix} (n-1)^3 & -(n-1)^2 \\ 1 & n-1 \end{pmatrix}$	$2(n-1)^3$
GM	$\begin{pmatrix} (n-1)^2 & -(n-1) \\ 1 & n-1 \end{pmatrix}$	$2(n-1)^2$
HM	$\begin{pmatrix} (n-1)^2 & -(n-1) \\ 1 & n-1 \end{pmatrix}$	$2(n-1)^2$

Proof. The Sum degree matrix of K_n is

$$SD(K_n) = \begin{pmatrix} 0 & 2(n-1) & 2(n-1) & \cdots & 2(n-1) \\ 2(n-1) & 0 & 2(n-1) & \cdots & 2(n-1) \\ 2(n-1) & 2(n-1) & 0 & \cdots & 2(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-1) & 2(n-1) & 2(n-1) & \cdots & 0 \end{pmatrix} = 2(n-1)(J_n - I_n)$$

where J_n is $n \times n$ matrix with all entries 1 and I_n is the identity matrix. The characteristic polynomial of $PD(K_n)$ is

$$\begin{aligned} f_m(G,\mu) &= \begin{vmatrix} \rho & -2(n-1) & -2(n-1) & \cdots & -2(n-1) \\ -2(n-1) & \rho & -2(n-1) & \cdots & -2(n-1) \\ -2(n-1) & -2(n-1) & \rho & \cdots & -2(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2(n-1) & -2(n-1) & -2(n-1) & \cdots & \rho \end{vmatrix} \\ &= (\rho + 2(n-1)^2) \begin{vmatrix} \rho & -2(n-1) & -2(n-1) & \cdots & -2(n-1) \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix} \\ &= (\rho + 2(n-1)^2)(\rho + 2(n-1))^{(n-1)} \end{aligned}$$

So, $Spec_{SD}(K_n) = \begin{pmatrix} 2(n-1)^2 & -2(n-1) \\ 1 & n-1 \end{pmatrix}$. So, Proof of calculation other central indices energy are similar to proof of Sum degree energy of complete graph. □

Let G be a graph on n vertices, $n \geq 1$. Then $CIE(G) = 0$ if and only if $G \cong \bar{K}_n$. Let $G = G_1 \cup G_2 \cup \cdots \cup G_m$. Then

(i) $E(G) = E(G_1) + E(G_2) + \cdots + E(G_m)$.

(ii) $GME(G) = GME(G_1) + GME(G_2) + \cdots + GME(G_m)$.

If the graph G is regular of degree k, $k > 0$, then $GME = kE(G)$. If, in addition $k = 0$, then $GME = 0$.

Proof. If $k = 0$, then $G \cong \bar{K}_n$. From Lemma 2, we know that $GME = 0$. Suppose now that G is regular of degree $k > 0$, that is $d_1 = d_2 = \cdots = d_n = k$. Then all non-zero terms in GM matrix are equal to k. This implies that $GM(G) = kA(G)$. Then we have $\rho_i = k\lambda_i$, and therefore $GME = kE(G)$. □

If G is the complete bipartite graph $K_{n,n}$, the spectral and energy of central indices are calculated as follow,

Matrix name	Spectral	$E_{CI}(G)$
A	$\begin{pmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{pmatrix}$	$2n$
SD	$\begin{pmatrix} 2n^2 & 0 & -2n^2 \\ 1 & 2n-2 & 1 \end{pmatrix}$	$4n^2$
AM	$\begin{pmatrix} n^2 & 0 & -n^2 \\ 1 & 2n-2 & 1 \end{pmatrix}$	$2n^2$
PD	$\begin{pmatrix} n^3 & 0 & -n^3 \\ 1 & 2n-2 & 1 \end{pmatrix}$	$2n^3$
GM	$\begin{pmatrix} n^2 & 0 & -n^2 \\ 1 & 2n-2 & 1 \end{pmatrix}$	$2n^2$
HM	$\begin{pmatrix} n^2 & 0 & -n^2 \\ 1 & 2n-2 & 1 \end{pmatrix}$	$2n^2$

Proof. The proof is similar to Theorem 2. □

If G is the complete bipartite graph $K_{n,m}$, with $n + m$ vertices. the spectral and energy of central indices are calculated as follow,

Matrix name	Spectral	$E_{CI}(G)$
A	$\begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$	$2\sqrt{mn}$
SD	$\begin{pmatrix} (m+n)\sqrt{mn} & 0 & -(m+n)\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$	$2(m+n)\sqrt{mn}$
AM	$\begin{pmatrix} \frac{(m+n)\sqrt{mn}}{2} & 0 & -\frac{(m+n)\sqrt{mn}}{2} \\ 1 & m+n-2 & 1 \end{pmatrix}$	$(m+n)\sqrt{mn}$
PD	$\begin{pmatrix} mn\sqrt{mn} & 0 & -mn\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$	$2mn\sqrt{mn}$
GM	$\begin{pmatrix} mn & 0 & -mn \\ 1 & m+n-2 & 1 \end{pmatrix}$	$2mn$
HM	$\begin{pmatrix} \frac{2mn\sqrt{mn}}{m+n} & 0 & -\frac{2mn\sqrt{mn}}{m+n} \\ 1 & m+n-2 & 1 \end{pmatrix}$	κ'

Where $\kappa' = \frac{4mn\sqrt{mn}}{m+n}$.

Proof. The proof is similar to Theorem 2. □

Let G be a bipartite compelete graph, we know number of edges equal mn then the Geometric mean energy is twice the number of edges, that is $GME = 2mn$. The crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and edge set $\{u_i v_i : 1 \leq i, j \leq n, i \neq j\}$. Therefore S_n^0 coincides with the complete bipartite graph $K_{n,n}$ with the horizontal edges removed. If G is the crown graph S_n^0 with $2n$ vertices, the Central Indices energy is defined

Matrix name	Spectral	$E_{CI}(G)$
A	$\begin{pmatrix} n-1 & 1 & -1 & -(n-1) \\ 1 & n-1 & n-1 & 1 \end{pmatrix}$	$2n$
SD	$\begin{pmatrix} 2(n-1)^2 & 2(n-1) & -2(n-1) & -2(n-1)^2 \\ 1 & n-1 & n-1 & 1 \end{pmatrix}$	$8(n-1)^2$
AM	$\begin{pmatrix} (n-1)^2 & n-1 & -(n-1) & -(n-1)^2 \\ 1 & n-1 & n-1 & 1 \end{pmatrix}$	$4(n-1)^2$
PD	$\begin{pmatrix} (n-1)^3 & (n-1)^2 & -(n-1)^2 & -(n-1)^3 \\ 1 & n-1 & n-1 & 1 \end{pmatrix}$	$4(n-1)^3$
GM	$\begin{pmatrix} (n-1)^2 & n-1 & -(n-1) & -(n-1)^2 \\ 1 & n-1 & n-1 & 1 \end{pmatrix}$	$4(n-1)^2$
HM	$\begin{pmatrix} (n-1)^2 & n-1 & -(n-1) & -(n-1)^2 \\ 1 & n-1 & n-1 & 1 \end{pmatrix}$	$4(n-1)^2$

If the graph G is regular of degree k, $k > 0$, then $PDE = k^2 E(G)$. If, in addition $k = 0$, then $PDE = 0$.

Proof. If $k = 0$, then $G \cong \bar{K}_n$. From Lemma 2, we know that $PDE = 0$. Suppose now that G is regular of degree $k > 0$, that is $d_1 = d_2 = \cdots = d_n = k^2$. Then all non-zero terms in GM matrix are equal to k. This implies that $PD(G) = k^2 A(G)$. Then we have $\rho_i = k^2 \lambda_i$, and therefore $PDE = kE(G)$. □

If the graph G is regular of degree k, $k > 0$, then

$$AME(G) = GME(G) = HME(G).$$

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