

## Large amplitude vibration prediction of rectangular plates by an optimal artificial neural network (ANN)

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### Abstract

In this paper, nonlinear equations of motion for laminated composite rectangular plates based on the first order shear deformation theory were derived. Using a perturbation method, the nonlinear equation of motion was solved and analytical relations were obtained for natural and nonlinear frequencies. After proving the validity of the obtained analytical relations, as an alternative and simple modeling technique, ANN was also employed to model the laminated rectangular plates and predict effects of different parameters on the natural and nonlinear frequencies of the plates. In this respect, an optimal ANN was selected and trained by training data sets obtained from analytical method and also tested by testing data sets. The obtained results were in good agreement with the analytical and published results.

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## 1. Introduction

Composite materials offer higher specific strength and stiffness than conventional materials. Relative lightness of composite materials enables the use of bigger sections that are inherently stiffer and stronger for bending and torsion, which is a considerable advantage for engineering structures. These materials offer outstanding fatigue and durability potentials and are very resistant to environmental conditions such as moisture, chemical attack, and high temperatures. To use these materials efficiently, it is necessary to develop appropriate models capable of accurately predicting their structural and dynamical behaviors [1, 2].

The Vibration of composite laminated rectangular plates has been extensively studied in the past years. Malik et al. [3] presented accurate three-dimensional elasticity solutions for the free linear vibrations of rectangular plates for some combinations of boundary conditions. Xiang et al. [4] focused for the first time on the free linear vibration analysis of laminated composite square plates using the trigonometric shear deformation theories. Wu et al. [5] used the mesh free least-squares-based finite difference (LSFD) method for solving large-amplitude free vibration problem of arbitrarily shaped thin plates. Finite element methods are widely used in studying linear and nonlinear vibration of rectangular laminates [6–

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18]. Some authors have used various numerical methods for studying the dynamical behavior of plates [19-23]. Many others have used analytical methods to study the vibration of plates [24-31].

Zhang et al. [32] used an artificial neural network to predict the dynamic mechanical properties of PTFE based short carbon fiber reinforced composites. Jodaei et al. [33] employed an optimal ANN method to model the functionally graded annular plates and predict the effects of different parameters on the natural frequency of the plates. They also used an artificial neural network to model the three-dimensional free vibration analysis of functionally graded piezoelectric annular plates [34]. Singh et al. [35] developed a regression based artificial neural network model to find the frequency of annular elliptic and circular plates. Gunes et al. [36, 37] investigated the free vibration behavior of an adhesively bonded functionally graded single lap joint using the finite element method and back-propagation artificial neural network method. Apalak and Yildirim [38] analyzed the first ten natural frequencies and mode shapes of the adhesive joint of an adhesively bonded composite single lap joint with unidirectional laminated narrow plates and subject to clamped-free condition. Rouss et al. [39] designed an experimental set to model a complex nonlinear mechanical system by a multilayer perceptron neural network. Reddy et al. [40] used an artificial neural network to predict the natural frequency of laminated composite plates under clamped boundary condition and applied some finite element software to obtain the required natural frequencies for training and testing the ANN model.

In this paper, the nonlinear equations of motion were derived for laminated composite rectangular plates. Anti-symmetric angle-ply and symmetric cross-ply composite plates were also considered. The boundary condition was taken to be movable simply-supported. Using a simple procedure compared with other analytical studies along with a perturbation method, the nonlinear equation of motion was solved and analytical relations were obtained

for natural and nonlinear frequencies. proving the validity of the obtained analytical relations, as an alternative and simple modeling technique, ANN was also employed to model the laminated rectangular plates and predict effects of different parameters on natural and nonlinear frequencies of the plates. In this respect, an optimal ANN was selected and trained by training data sets obtained from analytical method and also tested by testing data sets. The obtained results were in good agreement with the analytical method and published results.

## 2. Analytical solution

### 2.1. Solving the problem

Equations of motion of rectangular plates, based on the first order shear deformation theory are [41]:

$$N_{x,x} + N_{xy,y} = I_0 u_{0,tt} + I_1 \varphi_{x,tt} \quad (1)$$

$$N_{xy,x} + N_{y,y} = I_0 v_{0,tt} + I_1 \varphi_{y,tt} \quad (2)$$

$$Q_{x,x} + Q_{y,y} + N(w_0) + q = I_0 w_{0,tt} \quad (3)$$

$$M_{x,x} + M_{xy,y} - Q_x = I_2 \varphi_{x,tt} + I_1 u_{0,tt} \quad (4)$$

$$M_{xy,x} + M_{y,y} - Q_y = I_2 \varphi_{y,tt} + I_1 v_{0,tt} \quad (5)$$

where subscript ‘,’ denotes partial differentiation with respect to the following parameters,  $u_0$ ,  $v_0$ , and  $w_0$  are the displacements of a material point on the mid-surface along  $x$ -,  $y$ -, and  $z$ - axes, respectively,  $\varphi_x$  and  $\varphi_y$  are the rotations of a transverse normal about the  $y$ - and  $x$ -axes, respectively (Fig. 1),  $N_x$ ,  $N_y$ , and  $N_{xy}$  are the in-plane force resultants,  $Q_x$  and  $Q_y$  are the transverse force resultants,  $M_x$ ,  $M_y$ , and  $M_{xy}$  are the moments resultants,  $I_0$ ,  $I_1$ , and  $I_2$  are the mass moments of inertia,  $q$  is the applied transverse force which is zero in the free vibration, and  $N(w_0)$  is the only nonlinear term in the equations of motion and is in the following form:

$$N(w_0) = (N_x w_{0,x} + N_{xy} w_{0,y})_{,x} + (N_{xy} w_{0,x} + N_y w_{0,y})_{,y} \quad (6)$$

The boundary condition is taken to be movable simply-supported which is expressed by:

$$\begin{aligned} w = w_{,xx} = \psi_{,xy} = \int_0^b \psi_{,yy} dy &= 0 \quad (x=0, a) \\ w = w_{,yy} = \psi_{,xy} = \int_0^a \psi_{,xx} dx &= 0 \quad (y=0, b) \end{aligned} \quad (7)$$

Where  $a$  and  $b$  are the length and width of the plate, respectively, and  $\psi$  is the force function defined by:

$$N_x = \psi_{,yy} \quad N_y = \psi_{,xx} \quad N_{xy} = -\psi_{,xy} \quad (8)$$

Assuming the density of plate material ( $\rho_0$ ) as an even function of thickness ( $z$ ) and dealing with thin plates which makes it possible to neglect the in-plane inertia effects (i.e.  $u_{0,tt}$  and  $v_{0,tt}$ ), Eqs. (1-5) are reduced to the following equations, which are written in terms of the displacements and force function [25]:

$$\begin{aligned} KA_{55} [w_{,xx} + \varphi_{x,x}] + KA_{44} [w_{,yy} + \varphi_{y,y}] + \psi_{,yy} w_{,xx} \\ + \psi_{,xx} w_{,yy} - 2\psi_{,xy} w_{,xy} + q = I_0 \ddot{w} \end{aligned} \quad (9)$$

$$\begin{aligned} D_{11}^* \varphi_{x,xx} + D_{12}^* \varphi_{x,yy} + D_{66}^* (\varphi_{x,yy} + \varphi_{y,xy}) - KA_{55} [w_{,x} + \varphi_x] \\ + (B_{61}^* - B_{26}^*) \psi_{,xy} - B_{16}^* \psi_{,yy} = I_2 \ddot{\varphi}_x \end{aligned} \quad (10)$$

$$\begin{aligned} D_{12}^* \varphi_{x,xy} + D_{22}^* \varphi_{y,yy} + D_{66}^* (\varphi_{x,xy} + \varphi_{y,xx}) - \\ KA_{44} [w_{,y} + \varphi_y] + (B_{62}^* - B_{16}^*) \psi_{,xy} - B_{26}^* \psi_{,xx} = I_2 \ddot{\varphi}_y \end{aligned} \quad (11)$$

along with a compatibility equation in the following form [25, 42]:

$$\begin{aligned} A_{22}^* \psi_{,xxx} + (2A_{12}^* + A_{66}^*) \psi_{,xyy} + A_{11}^* \psi_{,yyy} = (2B_{26}^* - B_{61}^*) w_{,xyy} + \\ (2B_{16}^* - B_{62}^*) w_{,xyy} + w_{,xy}^2 - w_{,xx} w_{,yy} \end{aligned} \quad (12)$$

Where  $K$  is shear correction factor,  $A_{ij}$  are extensional stiffness,  $B_{ij}$  are bending-extension-coupling stiffness, and  $D_{ij}$  are bending stiffness. Unknown parameters of Eqs. (10-12) are obtained by [41- 43]:

$$\begin{aligned} \mathbf{A}^* = \mathbf{A}^{-1}, \quad \mathbf{B}^* = -\mathbf{A}^{-1} \mathbf{B}, \quad \mathbf{D}^* = \mathbf{D} - \mathbf{B} \mathbf{A}^{-1} \mathbf{B} \\ \mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & B_{26} \\ B_{16} & B_{26} & 0 \end{bmatrix}, \\ \mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}, \quad \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} = \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \end{aligned} \quad (13)$$

where, for the symmetric plates,  $\mathbf{B} = \mathbf{0}$  must be considered.

For the boundary condition of Eq. (7), a trial function for  $w$  can be assumed as:

$$w = hf(t) \sin(\pi x/a) \sin(\pi y/b) \quad (14)$$

Where  $h$  is thickness of the plate and  $f(t)$  is an unknown time function. So, the force function is obtained by substituting Eq. (14) for Eq. (12):

$$\psi = \frac{h^2 f^2(t)}{32} \left\{ \frac{a^2}{A_{22} b^2} \cos(2\pi x/a) + \frac{b^2}{A_{11} a^2} \cos(2\pi y/b) \right\} \quad (15)$$

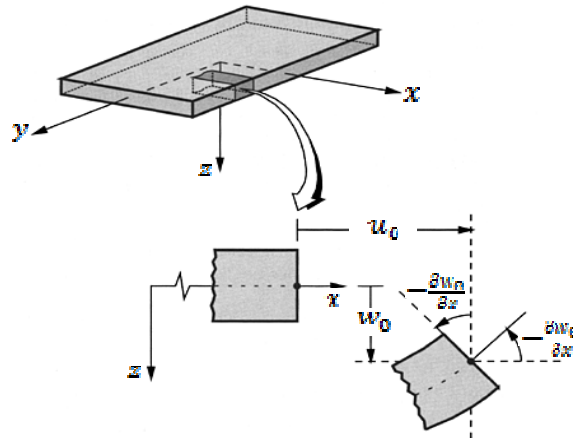
Equations (10 and 11) lead to a set of equations with two unknown parameters which are  $\varphi_x$  and  $\varphi_y$ . By solving this set of equations and substituting the obtained parameters (i.e.  $\varphi_x$  and  $\varphi_y$ ) along with  $\psi$  from Eq. (15) for Eq. (9), the following nonlinear partial differential equation is obtained ( $\varphi_x$  and  $\varphi_y$  are given in Appendix A):

$$\begin{aligned} K \left[ A_{55} \frac{\partial}{\partial x} (L_2^* L_6^* - L_3^* L_5^*) + A_{44} \frac{\partial}{\partial y} (L_3^* L_4^* - L_1^* L_6^*) \right] w \\ + K \left[ A_{55} \frac{\partial}{\partial x} (L_2^* L_8^* - L_5^* L_7^*) + A_{44} \frac{\partial}{\partial y} (L_4^* L_7^* - L_1^* L_8^*) \right] \psi \\ + \left\{ K \left[ A_{55} \frac{\partial^2}{\partial x^2} + A_{44} \frac{\partial^2}{\partial y^2} \right] + \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2}{\partial y^2} \right. \\ \left. - 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2}{\partial x \partial y} - I_0 \frac{\partial^2}{\partial t^2} \right\} \times (L_2^* L_4^* - L_1^* L_5^*) w = 0 \end{aligned} \quad (16)$$

Where  $L_i^*$  are partial differential operators given in Appendix A for the anti-symmetric angle-ply laminated rectangular plates.

The Galerkin method is applied by  $\iint_A L \cdot w dx dy = 0$ , in which  $A$  is area of the rectangular plate and  $L$  is the left-hand side of Eq. (16); it transforms the nonlinear PDE of Eq. (16) into a nonlinear ODE in terms of  $f(t)$ . Substitution of the dimensionless time  $\tau = \sqrt{A} t / a^2$  for this nonlinear ODE results in the following dimensionless nonlinear ODE:

$$f_{,\tau\tau} + \omega^2 f + \alpha^2 f^3 + \beta^2 f_{,\tau\tau} f^2 + \gamma^2 f_{,\tau}^2 f = 0 \quad (17)$$



**Fig. 1.** Generalized displacements of a rectangular plate in the first order shear deformation theory.

where  $\omega$  is dimensionless natural frequency,  $\alpha^2$  is coefficient of nonlinear stiffness term, and  $\beta^2$  and  $\gamma^2$  are coefficients of nonlinear inertia terms. In dimensionless time ( $\tau$ ),  $A = E_2 h^3 / I_0$ .

To solve Eq. (17) by a perturbation method, the nonlinear terms of Eq. (17) are multiplied by a small, dimensionless parameter ( $\varepsilon$ ) which is equated to unity after computations.  $\varepsilon$  is the order of the amplitude of motion and is used as a bookkeeping device in obtaining the approximate solution. So, the nonlinear terms of Eq. (17) are multiplied by  $\varepsilon$  to make the use of perturbation method possible:

$$f_{,\tau\tau} + \omega^2 f = -\varepsilon \left\{ \alpha^2 f^3 + \beta^2 f_{,\tau\tau} f^2 + \gamma^2 f_{,\tau}^2 f \right\} \quad (18)$$

According to Nayfeh and Mook [44], the independent time variables are defined by  $T_n = \varepsilon^n \tau$ , ( $n = 1, 2, \dots$ ), and  $f$  can be approximated by  $f(\tau, \varepsilon) = f_0(T_0, T_1) + \varepsilon f_1(T_0, T_1)$ . Derivatives with respect to  $\tau$  in Eq. (18) can be written in terms of partial derivatives of  $T_n$ . To this end, it should be noted that  $d/d\tau = D_0 + \varepsilon D_1$  and  $d^2/d\tau^2 = D_0^2 + 2\varepsilon D_0 D_1$ , where  $D_0$  and  $D_1$  denote  $\partial/\partial T_0$  and  $\partial/\partial T_1$ , respectively. Therefore, substitution of these relations for Eq. (18) gives:

$$D_0^2 f_0 + \omega^2 f_0 = 0 \quad (19)$$

$$D_0^2 f_1 + \omega^2 f_1 = -2D_0 D_1 f_0 - \alpha^2 f_0^3 - \beta^2 f_0^2 (D_0^2 f_0) - \gamma^2 f_0 (D_0 f_0)^2 \quad (20)$$

The solution of Eq. (19) is:

$$f_0 = X(T_1) \exp(i\omega T_0) + cc \quad (21)$$

where  $X$  is an unknown complex function of  $T_1$  and  $cc$  denotes the complex conjugate of preceding term.

To have a periodic solution after the substitution of Eq. (21) for Eq. (20), the following is required [44]:

$$-2i\omega X' - 3X^2 \bar{X} \alpha^2 + \omega^2 X^2 \bar{X} \beta^2 = 0 \quad (22)$$

which is called solvability condition. If Eq. (22) is not satisfied, instability occurs and the amplitude of motion grows over time.

By defining  $X$  in the polar form (i.e.,  $X = (r/2)\exp(is)$ ) and after its substitution in the above equation, the nonlinear frequency is obtained:

$$\omega_{NL} = \omega + \frac{r_0^2}{16\omega} \left[ 2r_1 + \left( \frac{r_2}{2} + \frac{r_3}{8} + r_4 \right) r_0^2 \right] \quad (23)$$

$$\frac{\omega_{NL}}{\omega} = \left[ 1 + \frac{r_1 r_0^2}{4\omega^2} + \frac{r_0^4}{8\omega^2} \left( \frac{r_1^2}{8\omega^2} + \frac{r_2}{2} + \frac{r_3}{8} + r_4 \right) \right]^{1/2}$$

where  $r_0$  is a constant parameter which shows the dimensionless initial displacement (i.e.,  $r_0 \equiv w_{\max}/h$ ). The unknown parameters of Eq. (23) are:

$$r_1 = 3\alpha^2 - \beta^2\omega^2, \quad r_2 = (\alpha^2 - 3\beta^2\omega^2)r_1/(8\omega^2),$$

$$r_3 = -r_1^2/\omega^2, \quad r_4 = -\beta^2 r_1 \quad (24)$$

## 2. 2. Validating the analytical solution

For a four-layered cross-ply square plate with the following material properties;  $E_1/E_2 = \text{open}$ ;  $G_{12} = G_{13} = 0.6E_2$ ;  $G_{23} = 0.5E_2$ ;  $\nu_{12} = 0.25$ , some dimensionless frequencies ( $\omega$ ) are obtained and compared with the results of Liew et al. [45] which is shown in Table 1. It can be observed that, for thinner plates and for plates with low orthotropy degrees ( $E_1/E_2$  ratios), the obtained frequencies are more accurate. Fundamental frequencies of a four-layered  $[45^\circ/-45^\circ/45^\circ/-45^\circ]$  composite plate are compared with some published results which are shown in Table 2. Material properties of the studied plate are:  $E_1 = 40E_2$ ,  $G_{12} = 0.6E_2$ ,  $\nu_{12} = 0.25$ ,  $G_{13} = G_{23} = 0.5E_2$ . Although the proposed analytical procedure is based on the first order shear deformation theory, according to Table 2, the results are in good agreement with the results of highly accurate higher order shear deformation theory (HSDT) presented by Alibeigloo et al. [46].

Nonlinear frequency ratio ( $\omega_{NL}/\omega$ ) of an isotropic square plate is also obtained which is shown in Table 3. Accordingly, the considered analytical approach gives good results for the nonlinear frequency as well. It is also observed that the nonlinear frequency is related to the amplitude of motion in a way that the increase of initial displacement results in higher nonlinear frequencies. So, the validity of the analytical solution is approved and subsequently Eqs. (17 and 23) are used to obtain the required data in training the neural network model.

**Table 1.** Dimensionless fundamental frequency of a symmetric laminated square plate.

$a/h$	Solutions	$E_1/E_2$		
		10	20	30
5	Liew et al. [45]	8.299	9.568	10.327
	Analytical	8.439	9.821	10.664
10	Reddy [41]	9.853	12.383	13.892
	Analytical	9.862	12.261	13.928

**Table 2.** Comparing the dimensionless fundamental frequency of a four-layered rectangular plate.

Method	$a/b$	1	2
Ghosh and Dey [12]	10	18.06	31.28
HSDT [46]		17.974	33.553
Analytical		18.044	33.496
Ghosh and Dey	30	23.28	50.89
HSDT		22.691	49.547
Analytical		22.675	49.644
Ghosh and Dey	50	23.91	53.68
HSDT		23.296	52.254
Analytical		23.210	52.155

**Table 3.** Nonlinear frequency ratio of isotropic square plates ( $a/h = 100$ ,  $\nu = 0.3$ ).

$r_0$	0.2	0.6	1.0
Singha and Daripa [17]	1.00516	1.04559	1.12239
Experimental [21]	1.00634	1.04502	1.11994
Analytical	1.00511	1.04525	1.12197

## 3. Neural network modeling

### 3. 1. Constructing the neural network model

The main advantage of ANN is its ability to model a problem using examples, rather than analytical description. Figure 2 shows the basic configuration of a neuron model.

An ANN model needs to be trained from an existing training set including many pairs of input–output elements. One of the most popular learning algorithms is the back-error propagation algorithm [47], which trains a neural network until the mean square errors between the training output data and the predicted output are minimized.

In this study, the proposed ANN model shown in Fig. 3 was trained by the input data obtained from Eqs. (17 and 23). The input layer needs random sets of data including  $E_1/E_2$ ,  $G_{12}/E_2$ ,  $G_{13}/E_2$ ,  $G_{23}/E_2$ ,  $a/b$ , and  $a/h$  ratios, number of layers ( $N$ ), and fiber orientation ( $\theta$ ) for each of the training seasons. Also, the output layer gives the predicted dimensionless natural frequency and nonlinear frequency ratio.

This configuration has 8–21–19–2 neurons in the input, hidden and output layers, respectively. The Levenberg–Marquardt back-propagation algorithm is used in a feed-

forward, two hidden layer network [36, 47]. The tangent sigmoid transfer function is used as

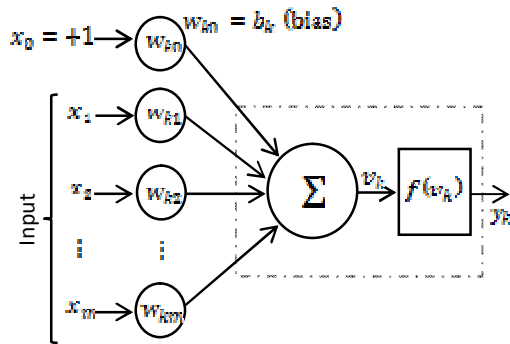


Fig. 2. A typical neuron model.

the transfer functions. The proposed neural network model was developed in MATLAB environment using the neural network toolbox. The input data set included 1200 patterns based on analytical solution developed in the previous section for a random set of design variables. Random 840 data patterns were used for the training of the neural network model and the remaining 360 patterns were randomly selected to use as validation and testing (180 patterns for each).

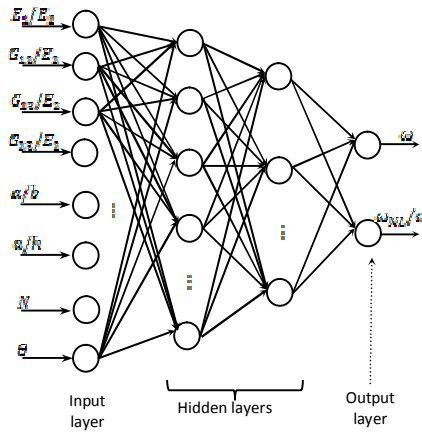


Fig. 3. ANN model of the present study.

### 3. 2. Results of the proposed ANN model

The database in this study included 840 data sets for different plate parameters obtained from the analytical analysis. Results of the

performance of the proposed 8–21–19–2 ANN model are given in Fig. 4, which shows the mean squared error of the network starting at a large value and decreasing to a smaller value. In other words, the network was learning. The plot had three lines, because the whole input and target vectors were randomly divided into three sets. Training on the training vectors continued as long as the training reduced the network's error on the validation vectors.

For a three-layered cross-ply square laminated plate with the following material properties;  $E_1 = 40E_2$ ,  $G_{12} = G_{13} = 0.6E_2$ ,  $G_{23} = 0.5E_2$ ,  $\nu_{12} = 0.25$ ,  $\omega$  and  $\omega_{NL}/\omega$  were obtained by the ANN model and the results were compared with those of the analytical and previously published ones, which are shown in Table 4 and Fig. 5, respectively.

In Table 5, frequencies of an eight-layered anti-symmetric square composite plate are given. The material properties were the same as the previous case and again the ANN model had

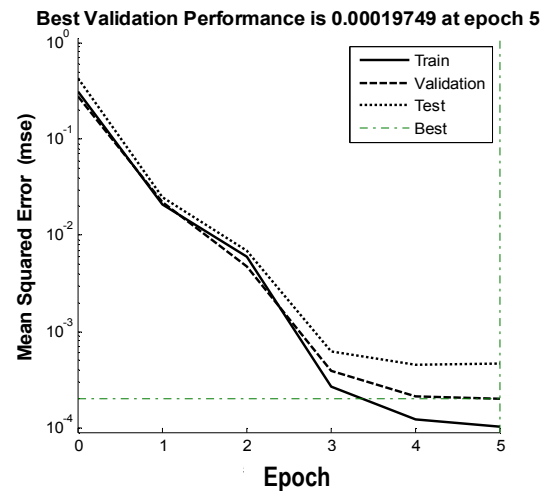


Fig. 4. Performance of the proposed ANN model.

good results compared with others in the literature and the analytical results. Tables 4 and 5 show that the increase of the length-to-thickness ratio increased dimensionless frequency, but decreased nonlinear frequency ratio of rectangular plates. It is observed in Fig. 5 that ANN model predicted the frequencies similar to Nguyen-Van et al. [48] and better than Ferreira et al. [49] which are both based on

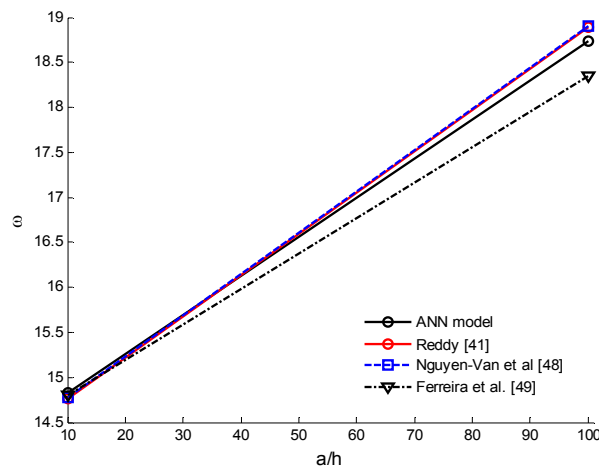
FSDT while the compared results in Table 5, i.e., Akhras and Li [50] and Reddy and Phan [51] are based on HSDT.

Effects of fiber orientation angle on the natural frequency were studied by the ANN model, the results of which are shown in Table 6 and compared with those of Reddy [41] and Xiang and Kang [52]. It can be observed that, by increasing the fiber orientation angle, the dimensionless frequency of the anti-symmetric rectangular plates increased, because for higher fiber orientation angles, the stiffness of the plate increased and in turn resulted in the increased

natural frequency of the plate.

#### 4. Conclusion

In this study, linear and nonlinear free vibrations of laminated rectangular plates were studied using an analytical method and comparative behavior modeling by ANN. The state variables included a combination of four moduli ratios, aspect ratio, length to thickness ratio, number of layers, and fibers orientation. The convergence and accuracy of the present



**Fig. 5.** Dimensionless frequencies of a three-layered  $[0^\circ/90^\circ/0^\circ]$  cross-ply square plate.

**Table 4.** Comparing the frequencies of a cross-ply  $[0^\circ/90^\circ/0^\circ]$  square plate predicted by ANN model with the analytical solution.

$a/h$	Model	$\omega$	$\omega_{NL}/\omega$
10	Analytical	14.8253	1.2417
	ANN model	14.8270	1.2337
20	Analytical	17.5108	1.1938
	ANN model	17.5396	1.1840
25	Analytical	17.9708	1.1853
	ANN model	17.9723	1.1751
50	Analytical	18.6474	1.1738
	ANN model	18.6552	1.1602
100	Analytical	18.8317	1.1589
	ANN model	18.7359	1.1638

method were validated by comparing the obtained results with those published in the literature. The results demonstrated that ANN can acceptably model the linear and nonlinear behaviors of laminated rectangular plates.

It was also seen that:

- Higher aspect ratios, length-to-thickness ratios, and moduli ratios increased dimensionless frequency of composite rectangular plates (see Tables 1 and 2).
- Unlike the linear frequency, nonlinear frequency was a function of the amplitude of motion which can be observed in Eq. (23) and Table 3.
- Length-to-thickness ratio had a minor effect on the nonlinear frequency ratio of composite rectangular plates (see Tables 4 and 5).
- Fiber orientation tended to increase the stiffness of the anti-symmetric rectangular plate, which was why dimensionless frequency of a plate with higher fiber orientation angle was higher (see Table 6).

**Table 5.** Frequencies of an eight-layered  $[45^\circ/-45^\circ/\dots]$  square laminated plate.

$a/h$	Model	$\omega$	$\omega_{NL}/\omega$
10	Akhras and Li [50]	19.264	-
	Reddy and Phan [51]	19.266	-
	Analytical	19.3422	1.0171
	ANN model	19.3190	1.0161
20	Akhras and Li	23.238	-
	Reddy and Phan	23.239	-
	Analytical	23.3135	1.0120
	ANN model	23.3007	1.0142
100	Akhras and Li	25.175	-
	Reddy and Phan	25.174	-
	Analytical	25.179	1.0104
	ANN model	25.1802	1.0111

**Table 6.** Dimensionless frequencies of a six-layered anti-symmetric square plate ( $a/h = 20$ ).

$\theta$	Method	$\omega$
30°	Reddy [41]	21.648
	Xiang and Kang [52]	22.2841
	ANN model	22.8018
45°	Reddy	22.877
	Xiang and Kang	23.6072
	ANN model	23.2616

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## Appendix A

$$L_1^* = (D_{66} + D_{12}) \frac{\partial^2}{\partial x \partial y}, L_3^* = KA_{44} \frac{\partial}{\partial y}$$

$$L_2^* = D_{66} \frac{\partial^2}{\partial x^2} + D_{22} \frac{\partial^2}{\partial y^2} - I_2 \frac{\partial^2}{\partial t^2} - KA_{44} \quad (A.1)$$

$$L_4^* = D_{11} \frac{\partial^2}{\partial x^2} + D_{66} \frac{\partial^2}{\partial y^2} - I_2 \frac{\partial^2}{\partial t^2} - KA_{55},$$

$$L_5^* = (D_{66} + D_{12}) \frac{\partial^2}{\partial x \partial y}, L_6^* = KA_{55} \frac{\partial}{\partial x} \quad (A.2)$$

$$L_7^* = (B_{16}^* - B_{62}^*) \frac{\partial^3}{\partial x \partial y^2} + B_{26}^* \frac{\partial^3}{\partial x^3},$$

$$L_8^* = (B_{26}^* - B_{61}^*) \frac{\partial^3}{\partial x^2 \partial y} + B_{16}^* \frac{\partial^3}{\partial y^3} \quad (A.3)$$

$$L_9^* = \frac{L_2^* L_6^* - L_3^* L_5^*}{L_2^* L_4^* - L_1^* L_5^*}, L_{10}^* = \frac{L_8^* L_2^* - L_1^* L_7^*}{L_2^* L_4^* - L_1^* L_5^*},$$

$$L_{11}^* = \frac{L_3^* L_4^* - L_5^* L_6^*}{L_2^* L_4^* - L_1^* L_5^*}, L_{12}^* = \frac{L_4^* L_7^* - L_5^* L_8^*}{L_2^* L_4^* - L_1^* L_5^*} \quad (A.4)$$

$$\varphi_x = L_9^* w + L_{10}^* \psi, \varphi_y = L_{11}^* w + L_{12}^* \psi \quad (A.5)$$

For the symmetric plate  $B_{ij}^* = 0$ .