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Research Paper

Counting formulas for weakly labelled plane tree-like structures

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Abstract. Block graphs have been enumerated by various authors. In this paper, plane tree-like structures in which the blocks are labelled with integers in the set $\{1,2,\ldots,k\}$ such that the labels of the blocks are non-decreasing from left to right are introduced. These tree-like structures are called weakly labelled k-plane tree-like structures herein. Using symbolic method, generating functions and application of Lagrange-Bürmann inversion, the structures are counted by number of vertices, blocks, occurrences of labels, root degree and label of the eldest/youngest block child of the root, number of leaves, forests and outdegree sequence.

Keywords.weakly labelled k-plane tree-like structure, block, degree, eldest block child, youngest block child, leaf, forest, outdegree sequence.
Mathematics Subject Classification (2020): 05C15,05C10.

1 Introduction

Tree-like structures (or block graphs) are graphs which are connected and do not form cycles i.e. they posses the properties of trees. These include Husimi graphs, cacti and oriented cacti. Various classes of tree-like structures such as labelled tree-like structures, plane tree-like structures, t-ary tree-like structures and non-crossing tree-like structures have been studied in literature as a means of generalizing trees. In this paper, we enumerate plane tree-like structures in which the blocks are assigned labels in a certain way according to various

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set parameters. Formally, a maximal subgraph of a tree-like structure which if removed disconnects the graph is referred to as a *block*. If all the blocks in a tree-like structure are complete graphs (respectively, cycles) then the structure is called a *Husimi graph* (respectively, *cactus*). In Figure 1, we have Husimi graph on 6 vertices. Since all the blocks in a tree are complete

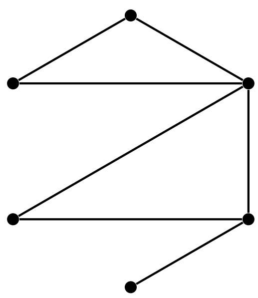

Figure 1. A Husimi graph on 6 vertices and three blocks of type $(1,2)$, i.e., one block with two vertices and two blocks with three vertices.

graphs with two vertices, then trees are Husimi graphs. Of course trees are not cacti since trees do not have cycles. Given a vertex v, the number of blocks in which v is one the vertices is the *degree* of v and the ordered arrangement of degrees of all vertices in the graph is the *degree sequence*. A vertex of degree 1 is referred to as a *leaf* and a vertex which is not a leaf is an *internal vertex*. A *forest* is a collection of tree-like structures. Husimi graphs were first studied in the context of statistical physics by Kodi Husimi [6] in 1950. Substantial amount of work has since been done by various authors to enumerate Husimi graphs by number of vertices, number of blocks and block types [5, 6, 10, 11]. The study on Husimi graphs was followed by introduction and enumeration of cacti according to number of vertices, blocks and block types by Harary and Uhlenbeck in [5] and Bóna, Bousquet, Labelle and Leroux in [2]. A cactus in which all the edges are oriented is an *oriented cactus*. Oriented cacti were introduced and enumerated by Springer in [21]. Based on the Prüfer correspondence for tree-like structures introduced by Springer in the aforementioned paper, Okoth [17] obtained a counting formula for labelled Husimi graphs with a given degree sequence.

In the paper [18], the author introduced *noncrossing tree-like structures* and enumerated them by number of vertices, blocks and block types. These are tree-like structures with vertices on the boundary of a circle such that blocks do not cross inside the circle. A depiction of a noncrossing Husimi graph is given in Figure 2.

In the same paper [18], Okoth introduced *plane tree-like structures* which are tree-like structures in which a given vertex has been earmarked as the root and all the blocks are ordered. If the blocks are complete graphs (respectively, cycles) then the structures are called *plane Husimi graphs* (respectively, *plane cacti*). Figure 3 is an illustration of a plane Husimi graph. The author found counting formulas for plane Husimi graphs and plane cacti by number of vertices, blocks, block types and number of leaves in [18]. Consider vertices u and v in

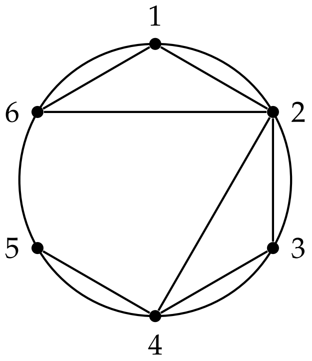

Figure 2. A noncrossing Husimi on 6 vertices with two blocks of type (1,2), i.e., one block with two vertices and two blocks with three vertices.

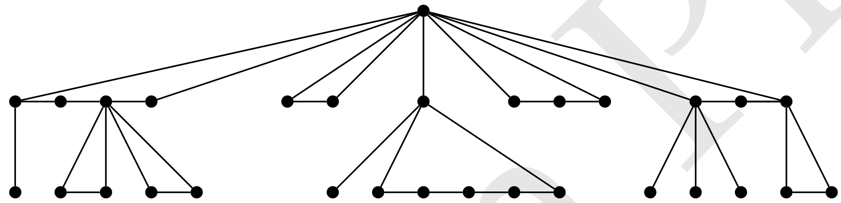

Figure 3. A plane Husimi graph on 30 vertices.

plane Husimi graph. Vertex u is a *child* (respectively, *parent*) of v if both u and v are in the same block but u lies on a lower level (respectively, higher level) than v. Moreover the block shared by u and v is a *block child* of v if v is a parent of u. For each internal vertex, the block child which is on the far left (respectively, right) is the *eldest block child* (respectively, *youngest block child*) of the internal vertex. *Outdegree* of a vertex is the number of block children of a vertex and the ordered arrangement of outdegrees of all vertices in a plane tree-like structure is the *outdegree sequence*. We remark that the outdegree and degree of the root coincide. Recently, Onyango, Okoth and Kasyoki [20] enumerated plane Husimi graphs by outdegree sequence, degree sequence and root degree. The authors used symbolic method to obtain the counting formulas. Before that, Kariuki and Okoth [7] had constructed bijections between the set of plane Husimi graphs and the sets of plane trees, dissections of regular polygons, sequences satisfying a certain condition, Dyck paths, standard Young tableaux among other combinatorial structures. Another category of tree-like structures that has been studied in literature is the set of *t-ary tree-like structures*. The authors of [20] defined t-ary tree-like structures as plane tree-like structures in which each internal vertex has no more than t block children. The authors then enumerated the structures using statistics such as number of vertices, blocks, block types, outdegree sequence and number of leaves. In Figure 3, we get a 5-ary Husimi graph. We note that enumeration of forests of tree-like structures is the subject of investigation in [15].

In 2025, Abayo and his co-authors [1] generalized plane tree-like structures by introducing and enumerating the set of k-dimensional plane tree-like structures which are plane tree-like structures with blocks assigned labels from the set $\{1,2,\ldots,k\}$ such that the labels of the block children of each internal non-root vertex are non-decreasing from left to right and the block children of the root are all labelled 1. The parameters of enumeration included blocks and block types, root degree, leaves, forests and outdegree sequence. In Figure 4, we get a 4-dimensional plane Husimi graph on 30 vertices with 14 blocks. If the block children of all

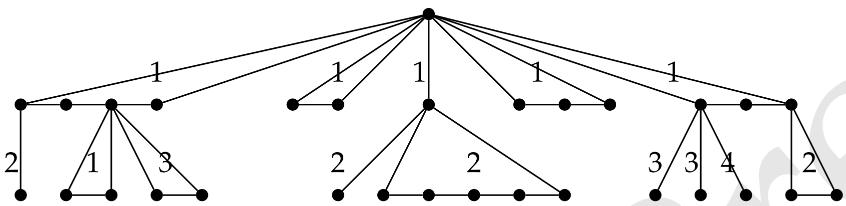

Figure 4. A 4-dimensional plane Husimi graph on 30 vertices.

internal vertices (including the root) in the plane tree-like structure are labelled with integers in the set $\{1,2,\ldots,k\}$ such that the labels are non-decreasing from left to right then we get *a weakly labelled k-plane tree-like structure*. Figure 5 gives a weakly labelled 4-plane Husimi graph with 30 vertices. We remark that if each block in a weakly labelled k-plane tree-like

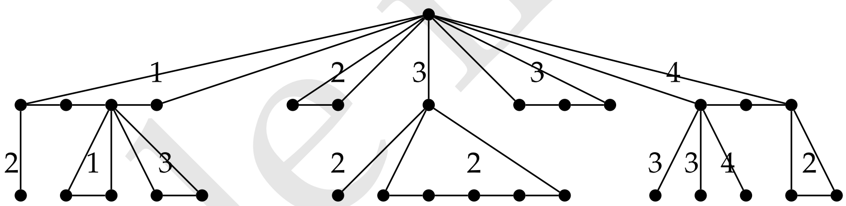

Figure 5. A weakly labelled 4-plane Husimi graph with 30 vertices.

structure has two vertices then we get a weakly labelled k-plane tree that was introduced and enumerated by Kariuki and Okoth in [8]. The results obtained in this paper thus generalize the results of the aforesaid paper. To obtain our results, we have used symbolic method and the following two versions of the Lagrange-Bürmann inversion formula.

Theorem 1.1 (Lagrange inversion formula, [22,23]). *Let $B(x)$ be a generating function that satisfies the functional equation $B(x) = x\phi(B(x))$, where $\phi(0) \neq 0$. Then, $n[x^n]B(x)^k = k[b^{n-1}]\phi(b)^n$.*

Theorem 1.2 (Lagrange-Bürmann inversion, [22, 23]). *Let $B(x)$ be a generating function that satisfies the functional equation $B(x) = x\phi(B(x))$, where $\phi(0) \neq 0$. Then*

$$n[x^n]A(B(x)) = [a^{n-1}]A'(a)\phi(a)^n,$$

where A is any arbitrary analytic function.

This paper is organized as follows. In Section 2, we enumerate the set of weakly labelled k-plane tree-like structures by number of blocks and occurrences of labels of a given type. The work is extended to enumerate the structures by root degree and label of the eldest block child of the root in Section 2. Sections 3 and 4 are set aside for enumeration of tree-like structures with a given number of leaves and forests with a prescribed number of components respectively. Outdegree sequence is the statistic of enumeration in Section 6. The study is concluded in Section 7 with research problems exposed therein.

2 Number of blocks and occurrences of labels

In the sequel, we get the main result of this section.

Theorem 2.1. *The number of weakly labelled k-plane Husimi graphs on n vertices with b blocks, n_i of which are of size i and b_j blocks are labelled j is given by*

$$\frac{1}{n}\prod_{j=1}^{k}\binom{n+b_j-1}{b_j}\frac{b!}{n_1!n_2!\cdots}. \tag{1}$$

Proof. Let $P(x,v_1,\ldots,v_k) = P$ be the multivariate generating function for weakly labelled k-plane Husimi graphs where x marks a vertex and v_j marks a block labelled j for $j = 1,\ldots,k$. Let w_i marks a block with $i-1$ vertices. Since $P(x,v_1,\ldots,v_k)$ consists of a root vertex and a sequence of plane Husimi graphs with blocks labelled 1, then those labelled 2, and so on, until we get a sequence of plane Husimi graphs with blocks labelled k, then we have

$$\begin{aligned} P &= x\cdot\frac{1}{1-v_1\sum_{j\geq 1}w_{i+1}P^i}\cdot\frac{1}{1-v_2\sum_{i\geq 1}w_{i+1}P^i}\cdots\frac{1}{1-v_k\sum_{i\geq 1}w_{i+1}P^i} \\ &= x\prod_{j=1}^{k}\left(1-v_j\sum_{i\geq 1}w_{i+1}P^i\right)^{-1}. \end{aligned}$$

By Lagrange inversion formula (Theorem 1.1), we get

$$\begin{aligned}
[x^n v_1^{b_1}\cdots v_k^{b_k}]P &= \frac{1}{n}[p^{n-1}v_1^{b_1}\cdots v_k^{b_k}]\prod_{j=1}^{k}(1-v_j\sum_{i\geq 1}w_{i+1}p^i)^{-n}\\
&= \frac{1}{n}[p^{n-1}v_1^{b_1}\cdots v_k^{b_k}]\prod_{j=1}^{k}\sum_{b\geq 0}\binom{-n}{b}(-v_j\sum_{i\geq 1}w_{i+1}p^i)^{b}\\
&= \frac{1}{n}[p^{n-1}v_1^{b_1}\cdots v_k^{b_k}]\prod_{j=1}^{k}\sum_{b\geq 0}\binom{n+b-1}{b}\left(\sum_{i\geq 1}w_{i+1}p^i\right)^{b}v_j^b\\
&= \frac{1}{n}[p^{n-1}]\prod_{j=1}^{k}\binom{n+b_j-1}{b_j}\left(\sum_{i\geq 1}w_{i+1}p^i\right)^{b}\\
&= \frac{1}{n}[p^{n-1}]\prod_{j=1}^{k}\binom{n+b_j-1}{b_j}\sum_{n_2+n_3+\cdots=b}\frac{b!p^{n_2+2n_3+\cdots}w_2^{n_2}w_3^{n_3}\cdots}{n_1!n_2!\cdots}\\
&= \frac{1}{n}\prod_{j=1}^{k}\binom{n+b_j-1}{b_j}\sum_{\substack{n_2+n_3+\cdots=b\\ n_2+2n_3+\cdots=n-1}}\frac{b!}{n_1!n_2!\cdots}.
\end{aligned}$$

This completes the proof. ☐

Setting $k=1$ in (1), then $b_1=b$ and we find that there are

$$\frac{1}{n}\binom{n+b-1}{b}\frac{b!}{n_1!n_2!\cdots}$$

plane Husimi graphs on n vertices with b blocks, n_j of which have size $j\geq 2$.

Corollary 2.2. *There are*

$$\frac{1}{n}\binom{kn+b-1}{b}\frac{b!}{n_1!n_2!\cdots}$$

weakly labelled k-plane Husimi graphs on n vertices with b blocks, n_j of which are of size $j\geq 2$.

Proof. We sum over all values of b_j in (1), i.e.,

$$\begin{aligned}
[x^n]P_i &= \sum_{b_1+\cdots+b_k=b}\frac{1}{n}\prod_{j=1}^{k}\binom{n+b_j-1}{b_j}\frac{b!}{n_1!n_2!\cdots}\\
&= \frac{1}{n}\sum_{b_1+\cdots+b_k=b}\prod_{j=1}^{k}\binom{-n}{b_j}(-1)^{b_j}\frac{b!}{n_1!n_2!\cdots}\\
&= \frac{1}{n}\sum_{b_1+\cdots+b_k=b}(-1)^{b_1+\cdots+b_k}\prod_{j=1}^{k}\binom{-n}{b_j}\frac{b!}{n_1!n_2!\cdots},
\end{aligned}$$

which can be written as

$$\begin{aligned}[x^n]P_i &= \frac{1}{n}(-1)^b\binom{-kn}{b}\frac{b!}{n_1!n_2!\cdots} \\ &= \frac{1}{n}\binom{kn+b-1}{b}\frac{b!}{n_1!n_2!\cdots}.\end{aligned}$$

□

Alternative proof of Corollary 2.2. Let $P(x)$ be the generating function for weakly labelled k-plane Husimi graphs where x marks a vertex. Let w_{i+1} marks a block of size i. Then

$$P = x \cdot \frac{1}{\left(1 - \sum_{j\geq 1} w_{i+1}P^i\right)^k}.$$

By Lagrange inversion formula (Theorem 1.1), we get

$$\begin{aligned}[x^n]P &= \frac{1}{n}[p^{n-1}]\left(1 - \sum_{i\geq 1} w_{i+1}p^i\right)^{-kn} \\ &= \frac{1}{n}[p^{n-1}]\sum_{b\geq 0}\binom{-kn}{b}\left(-\sum_{i\geq 1} w_{i+1}p^i\right)^b \\ &= \frac{1}{n}[p^{n-1}]\sum_{b\geq 0}\binom{kn+b-1}{b}\left(\sum_{i\geq 1} w_{i+1}p^i\right)^b \\ &= \frac{1}{n}[p^{n-1}]\sum_{b\geq 0}\binom{kn+b-1}{b}\sum_{n_2+n_3+\cdots=b}\frac{b!p^{n_2+2n_3+\cdots}w_2^{n_2}w_3^{n_3}\cdots}{n_1!n_2!\cdots} \\ &= \frac{1}{n}\sum_{b\geq 0}\binom{kn+b-1}{b}\sum_{\substack{n_2+n_3+\cdots=b \\ n_2+2n_3+\cdots=n-1}}\frac{b!}{n_1!n_2!\cdots}.\end{aligned}$$

Thus the proof. □

This result was proved by Onyango, Okoth and Kasyoki in [20].

Lemma 2.3 ([20]). *Let $n, b \geq 1$ and let $n_1, n_2, \ldots$ be non-negative integers such that $n_1 + n_2 + \cdots = b$ and $n_1 + 2n_2 + \cdots = p$. Then,*

$$\sum_{\substack{n_1+n_2+\cdots=b \\ n_1+2n_2+\cdots=p \\ n_1,n_2,\ldots\geq 0}}\frac{b!}{n_1!n_2!\cdots} = \binom{p-1}{b-1}.$$

We obtain the following corollary by summing over all values of n_j in (1), making use of Lemma 2.3.

Corollary 2.4. *The number of weakly labelled k-plane Husimi graphs on n vertices with b blocks such that b_j of the blocks are labelled j is given by*

$$\frac{1}{n}\prod_{j=1}^{k}\binom{n+b_j-1}{b_j}\binom{n-2}{b-1}. \tag{2}$$

Moreover, summing over all values of b_j in (2) as was done in the proof of Corollary 2.2, we get that there are

$$\frac{1}{n}\binom{kn+b-1}{b}\binom{n-2}{b-1} \tag{3}$$

weakly labelled k-plane Husimi graphs on n vertices with b blocks. We find the formula for the total number of weakly labelled k-plane Husimi graphs on n vertices by summing over all values of b, i.e. from 1 to $n-1$ in (3). The following result was obtained by Kariuki and Okoth in [8].

Corollary 2.5. *There are*

$$\frac{1}{n}\binom{(k+1)n-2}{n-1}$$

weakly labelled k-plane trees on n vertices.

Proof. Set $b=n-1$ in (3) and noting that the label of the edge in the plane tree can be assigned to its endpoint on the lower level. □

Remark 2.6. *There is only one way of converting a complete graph with at least* 3 *vertices into a cycle, so the formulas obtained here for plane Husimi graphs also count cacti.*

3 Root degree and label of the eldest block child of the root

In this section, we are interested in the formulas that count weakly labelled k-plane Husimi graphs given root degree and the label of the eldest block child of the root.

Lemma 3.1. *There are*

$$\frac{d}{n-1}\binom{k(n-1)+b-d-1}{b-d}\binom{n-1}{b} \tag{4}$$

weakly labelled k-plane Husimi graphs on n vertices with root of degree d such that all the d block children of the root are labelled $1\leq i\leq k$.

Proof. We use Lagrange-Bürmann inversion (Theorem 1.2) to extract the coefficient of x^n in $x\left(\sum_{i\geq 1} w_{i+1}P^i\right)^d$:

$$\begin{aligned}[x^n]x\left(\sum_{i\geq 1} w_{i+1}P^i\right)^d &= [x^{n-1}]\left(\sum_{i\geq 1} w_{i+1}P^i\right)^d\\ &= \frac{1}{n-1}[p^{n-2}]d\left(\sum_{i\geq 1} w_{i+1}p^i\right)^{d-1}\left(\sum_{i\geq 1} w_{i+1}ip^{i-1}\right)\left(\frac{x}{1-\sum_{i\geq 1} w_{i+1}p^i}\right)^{k(n-1)}.\end{aligned}$$

Binomial theorem gives

$$
\begin{aligned}
&[x^n]x\left(\sum_{i\geq 1} w_{i+1}P^i\right)^d \\
&= \frac{d}{n-1}[p^{n-2}]\left(\sum_{i\geq 1} w_{i+1}p^i\right)^{d-1}\left(\sum_{i\geq 1} w_{i+1}ip^{i-1}\right)\sum_{a\geq 0}\binom{k(n-1)+a-1}{a}\left(\sum_{i\geq 1} w_{i+1}p^i\right)^a \\
&= \frac{d}{n-1}[p^{n-2}]\sum_{a\geq 0}\binom{k(n-1)+a-1}{a}\left(\sum_{i\geq 1} w_{i+1}p^i\right)^{d+a-1}\left(\sum_{i\geq 1} w_{i+1}ip^{i-1}\right) \\
&= \frac{d}{n-1}[p^{n-2}]\sum_{a\geq 0}\binom{k(n-1)+a-1}{a}\frac{\mathrm{d}}{\mathrm{d}p}\frac{1}{d+a}\left(\sum_{i\geq 1} w_{i+1}p^i\right)^{d+a} \\
&= \frac{d}{n-1}\sum_{a\geq 0}\binom{k(n-1)+a-1}{a}(n-1)[p^{n-1}]\frac{1}{d+a}\left(\sum_{i\geq 1} w_{i+1}p^i\right)^{d+a} \\
&= \sum_{a\geq 0}\frac{d}{d+a}\binom{k(n-1)+a-1}{a}[p^{n-1}]\left(\sum_{i\geq 1} w_{i+1}p^i\right)^{d+a}.
\end{aligned}
$$

So

$$
[x^n]x\left(\sum_{i\geq 1} w_{i+1}P^i\right)^d = \sum_{a\geq 0}\frac{d}{d+a}\binom{k(n-1)+a-1}{a}\sum_{\substack{n_2+n_3+\cdots=d+a\\ n_2+2n_3+\cdots=n-1}}\frac{(d+a)!w_2^{n_2}w_3^{n_3}\cdots}{n_2!n_3!\dots}.
$$

The number of weakly labelled k-plane Husimi graphs on n vertices with b blocks and root degree d such that there are n_j blocks of size j and all the block children of the root are labelled i is given by

$$
d\binom{k(n-1)+b-d-1}{b-d}\frac{(b-1)!}{n_2!n_3!\dots},
$$

where $n_2+n_3+\cdots=b$. By Lemma 2.3, the number of weakly labelled k-plane Husimi graphs on n vertices with b blocks and root degree d is

$$
\begin{aligned}
d\binom{k(n-1)+b-d-1}{b-d}&\sum_{\substack{n_2+n_3+\cdots=b\\ n_2+2n_3+\cdots=n-1}}\frac{(b-1)!}{n_2!n_3!\dots} \\
&= \frac{d}{b}\binom{k(n-1)+b-d-1}{b-d}\sum_{\substack{n_2+n_3+\cdots=b\\ n_2+2n_3+\cdots=n-1}}\frac{b!}{n_2!n_3!\dots} \\
&= \frac{d}{b}\binom{k(n-1)+b-d-1}{b-d}\binom{n-2}{b-1}.
\end{aligned}
$$

□

On setting $k = 1$ in (4), we obtain the formula for the number of plane Husimi graphs with n vertices such that the root is of degree d, i.e.,

$$\frac{d}{n-1}\binom{n+b-d-2}{b-d}\binom{n-1}{b}. \tag{5}$$

Formula (5) was first discovered by Onyango, Okoth and Kasyoki in [20]. We also note that upon setting $b = n-1$ in (5), we obtain the formula for plane trees on n vertices with root degree d as

$$\frac{d}{n-1}\binom{2n-d-3}{b-d}$$

which was also obtained in [3]. By setting $k = 2$ and $b = n-1$ in (4), it follows that there are

$$\frac{d}{n-1}\binom{3n-d-4}{b-d}$$

noncrossing trees on n vertices with root degree d initially obtained by Noy in [12]. Summing over all values of d in (4), we get that there are

$$\frac{1}{n-1}\binom{k(n-1)+b}{b-1}\binom{n-1}{b} \tag{6}$$

weakly labelled k-plane Husimi graphs on n vertices with b blocks such that all the block children of the root are labelled i and on further setting $b = n-1$, it implies that there are

$$\frac{1}{n-1}\binom{(k+1)(n-1)}{n-2}$$

weakly labelled k-plane trees in which all the children of the root are labelled by a fixed integer i.

Theorem 3.2. *The number of weakly labelled k-plane Husimi graphs on n vertices with b blocks and root degree d such that the eldest block child of the root is labelled i is given by*

$$\frac{d}{n-1}\binom{k(n-1)+b-d-1}{b-d}\binom{n-1}{b}\binom{k-i+d-1}{d-1}. \tag{7}$$

Proof. There are

$$\frac{d}{n-1}\binom{k(n-1)+b-d-1}{b-d}\binom{n-1}{b}$$

weakly labelled k-plane Husimi graphs on n vertices with b blocks such that all the d block children of the root are labelled i (Lemma 3.1). Now, if the eldest block child of the root is to retain label i, then the siblings of the eldest block child of the root must be relabelled so that the labels are weakly increasing i.e., we start with a sequence of siblings labelled i on the far left, followed by a sequence of siblings labelled $i+1$ and so on until a sequence of siblings labelled k, on the far right. Since the root degree is d, then this is the same as choosing $d-1$ objects from a set with $k-i+1$ elements such that repetitions is allowed. The number of ways of doing this is

$$\binom{k-i+d-1}{d-1}.$$

The formula thus follows. □

Summing over all values of i in (7), we obtain the following result.

Corollary 3.3. *There are*

$$\frac{d}{n-1}\binom{k(n-1)+b-d-1}{b-d}\binom{n-1}{b}\binom{k+d-1}{d} \tag{8}$$

weakly labelled k-plane Husimi graphs on n vertices with b blocks and root degree d.

The following identity is proven in [9, Corollary 5.5].

Identity 3.4. *Let k_1 and k_2 be positive integers and m a nonnegative integer, then*

$$\sum_{a=0}^{m}\binom{k_1+a-1}{a}\binom{k_2+m-a-1}{m-a}=\binom{k_1+k_2+m-1}{m}.$$

If we sum over all d in (8), making of using Identity 3.4, we obtain

$$\frac{k}{n-1}\binom{kn+b-1}{b-1}\binom{n-1}{b}$$

as the number of weakly labelled k-plane Husimi graphs on n vertices with b blocks. This formula was already obtained in (3). Upon setting $i=k-i+1$ in (7), we obtain the following corollary.

Corollary 3.5. *The number of weakly labelled k-plane Husimi graphs with n vertices, b blocks and root of degree d such that the youngest block child of the root is labelled i is given by*

$$\frac{d}{n-1}\binom{k(n-1)+b-d-1}{b-d}\binom{n-1}{b}\binom{i+d-2}{d-1}. \tag{9}$$

If we sum over all values of d in (7), again making use of Identity (3.4), we get at the following result.

Corollary 3.6. *The number of weakly labelled k-plane Husimi graphs with n vertices and b blocks such that the eldest child of the root is labelled i is*

$$\frac{k(n-1)+b(k-i+1)}{(n-1)(kn-i+1)}\binom{kn+b-i-1}{b-1}\binom{n-1}{b}. \tag{10}$$

By setting $i = k$ in (10), we arrive at the following result.

Corollary 3.7. *There are*

$$\frac{1}{n-1}\binom{k(n-1)+b}{b-1}\binom{n-1}{b} \tag{11}$$

weakly labelled k-plane Husimi graphs on n vertices with b blocks such that the eldest block child of the root is labelled k. Note that all other block children of the root must be labelled k.

We remark that (11) was already obtained in (6).

Corollary 3.8. *The number of weakly labelled k-plane Husimi graphs on n vertices with b block such that the eldest block child of the root is labelled* 1 *is*

$$\frac{n+b-1}{bn}\binom{kn+b-2}{b-1}\binom{n-2}{b-1}. \tag{12}$$

Proof. Set $i = 1$ in (10). □

Setting $b = n-1$ in (12), we get that:

Corollary 3.9 ([8]). *There are*

$$\frac{2}{n}\binom{(k+1)n-3}{n-2}$$

weakly labelled k-plane trees on n vertices such that the eldest child of the root is labelled 1.

Summing over all values of d in (9), we get the following result.

Corollary 3.10. *The number of weakly labelled k-plane Husimi graphs on n vertices with b blocks such that the youngest child of the root is labelled i is*

$$\frac{k(n-1)+bi}{(n-1)(k(n-1)+i)}\binom{k(n-1)+b+i-2}{b-1}\binom{n-1}{b}. \tag{13}$$

Setting $i = k$ in (13), we find that (12) also gives the number of weakly labelled k-plane Husimi graphs on n vertices with b blocks such that the youngest block child of the root is labelled k. In addition, setting $i = 1$ in (13) reveals that (11) also counts weakly labelled k-plane Husimi graphs on n vertices with b blocks such that the youngest block child of the root is labelled 1. Consequently, we get the following corollary upon letting $i = 1$ and $b = n-1$ in (13).

Corollary 3.11. *There are*

$$\frac{1}{n-1)}\binom{(k+1)(n-1)}{n-2}$$

weakly labelled k-plane trees on n vertices such that the youngest child of the root is labelled 1.

On setting $i = k$ and $b = n-1$ in (13), we have the following result.

Corollary 3.12. *There are*

$$\frac{2}{n}\binom{k(n-1)-3}{n-2}$$

weakly labelled k-plane trees on n vertices such that the youngest child of the root is labelled k.

4 Leaves

This section is devoted to enumeration of weakly labelled k-plane Husimi graphs by number of leaves.

Theorem 4.1. *There are*

$$\frac{1}{n}\binom{n}{\ell}\binom{n-2}{b-1}\sum_{s\geq 0}\binom{n-\ell}{s}(-1)^s\prod_{j=1}^{k}\binom{n-\ell-s+b_j-1}{b_j} \tag{14}$$

weakly labelled k-plane Husimi graphs on n vertices with ℓ leaves and b blocks such that b_j blocks are labelled j.

Proof. Let $P(x,u,v_1,\ldots,v_k)=P$ be the multivariate generating function for weakly labelled k-plane Husimi graphs where x marks a vertex, u marks a leaf and v_j marks a vertex labelled j for $j=1,2,\ldots,k$. Also, let w_{i+1} denotes a block of size i. So,

$$\begin{aligned}P(x,u)&=xu-x+x\cdot\frac{1}{1-v_1\sum_{i\geq 1}w_{i+1}P^i}\cdot\frac{1}{1-v_2\sum_{i\geq 1}w_{i+1}P^i}\cdots\frac{1}{1-v_k\sum_{i\geq 1}w_{i+1}P^i}\\&=xu-x+x\frac{1}{\prod_{j=1}^{k}(1-v_j\sum_{i\geq 1}w_{i+1}P^i)}\\&=x\left(u+\frac{1-\prod_{j=1}^{k}(1-v_j\sum_{i\geq 1}w_{i+1}P^i)}{\prod_{j=1}^{k}(1-v_j\sum_{i\geq 1}w_{i+1}P^i)}\right).\end{aligned}$$

Lagrange inversion formula (Theorem 1.1) gives,

$$\begin{aligned}[x^n u^\ell v_1^{b_1}\cdots v_k^{b_k}]P&=\frac{1}{n}[p^{n-1}u^\ell v_1^{b_1}\cdots v_k^{b_k}]\left(u+\frac{1-\prod_{j=1}^{k}(1-v_j\sum_{i\geq 1}w_{i+1}p^i)}{\prod_{j=1}^{k}(1-v_j\sum_{i\geq 1}w_{i+1}p^i)}\right)^n\\&=\frac{1}{n}[p^{n-1}u^\ell v_1^{b_1}\cdots v_k^{b_k}]\sum_{a\geq 0}\binom{n}{a}u^a\left(\frac{1-\prod_{j=1}^{k}(1-v_j\sum_{i\geq 1}w_{i+1}p^i)}{\prod_{j=1}^{k}(1-v_j\sum_{i\geq 1}w_{i+1}p^i)}\right)^{n-a}\\&=\frac{1}{n}[p^{n-1}v_1^{b_1}\cdots v_k^{b_k}]\binom{n}{\ell}\left(1-\prod_{j=1}^{k}\left(1-v_j\sum_{i\geq 1}w_{i+1}p^i\right)\right)^{n-\ell}\prod_{j=1}^{k}\left(1-v_j\sum_{i\geq 1}w_{i+1}p^i\right)^{-(n-\ell)}.\end{aligned}$$

It follows that

$$
\begin{aligned}
[x^n u^\ell v_1^{b_1}\cdots v_k^{b_k}]P &= \frac{1}{n}[p^{n-1}v_1^{b_1}\cdots v_k^{b_k}]\binom{n}{\ell}\sum_{s\geq 0}\binom{n-\ell}{s}(-1)^s\prod_{j=1}^{k}\left(1-v_j\sum_{i\geq 1}w_{i+1}p^i\right)^{s-(n-\ell)} \\
&= \frac{1}{n}[p^{n-1}v_1^{b_1}\cdots v_k^{b_k}]\binom{n}{\ell}\sum_{s\geq 0}\binom{n-\ell}{s}(-1)^s\prod_{j=1}^{k}\sum_{b\geq 0}\binom{-(n-\ell-s)}{b}\left(-\sum_{i\geq 1}w_{i+1}p^i\right)^b v_j^b \\
&= \frac{1}{n}[p^{n-1}v_1^{b_1}\cdots v_k^{b_k}]\binom{n}{\ell}\sum_{s\geq 0}\binom{n-\ell}{s}(-1)^s\prod_{j=1}^{k}\sum_{b\geq 0}\binom{n-\ell-s+b-1}{b}\left(\sum_{i\geq 1}w_{i+1}p^i\right)^b v_j^b \\
&= \frac{1}{n}[p^{n-1}v_1^{b_1}\cdots v_k^{b_k}]\binom{n}{\ell}\sum_{s\geq 0}\binom{n-\ell}{s}(-1)^s \\
&\quad \prod_{j=1}^{k}\sum_{b\geq 0}\sum_{n_2+n_3+\cdots=b}\binom{n-\ell-s+b-1}{b}\frac{b!p^{n_2+2n_3+\cdots}w_2^{n_2}w_3^{n_3}\cdots}{n_2!n_3!\cdots}v_j^b \\
&= \frac{1}{n}\binom{n}{\ell}\sum_{s\geq 0}\binom{n-\ell}{s}(-1)^s\prod_{j=1}^{k}\binom{n-\ell-s+b_j-1}{b_j}\sum_{\substack{n_2+n_3+\cdots=b\\ n_2+2n_3+\cdots=n-1}}\frac{b!}{n_2!n_3!\cdots}.
\end{aligned}
$$

Summing over all values of n_j's making use of Lemma 2.3, we obtain

$$
[x^n u^\ell v_1^{b_1}\cdots v_k^{b_k}]P = \frac{1}{n}\binom{n}{\ell}\sum_{s\geq 0}\binom{n-\ell}{s}(-1)^s\prod_{j=1}^{k}\binom{n-\ell-s+b_j-1}{b_j}\binom{n-2}{b-1}.
$$

□

This following corollary is obtained by summing over all values of b_j in (14). We provide an alternative proof though by means of generating functions.

Corollary 4.2. *There are*

$$
\frac{1}{n}\binom{n}{\ell}\binom{n-2}{b-1}\sum_{a\geq 0}\binom{n-\ell}{a}\binom{k(n-\ell-a)+b-1}{b}(-1)^a \tag{15}
$$

weakly labelled k-plane Husimi graphs on n vertices with ℓ leaves and b blocks.

Proof. Let $P(x,u)$ be the bivariate generating function for weakly labelled k-plane Husimi graphs where x and u marks a vertex and a leaf respectively. Also let w_{i+1} denotes a block of size i. It follows that,

$$
P(x,u) = xu - x + x\cdot\frac{1}{\left(1-\sum_{i\geq 1}w_{i+1}P^i\right)^k} = x\left(u+\frac{1-\left(1-\sum_{i\geq 1}w_{i+1}P^i\right)^k}{\left(1-\sum_{i\geq 1}w_{i+1}P^i\right)^k}\right).
$$

By Lagrange inversion formula (Theorem 1.1), we get

$$\begin{aligned}
[x^n u^\ell]P =& \frac{1}{n}[p^{n-1}u^\ell]\left(u + \frac{1-\left(1-\sum_{i\geq 1} w_{i+1}p^i\right)^k}{\left(1-\sum_{i\geq 1} w_{i+1}p^i\right)^k}\right)^n \\
=& \frac{1}{n}[p^{n-1}u^\ell]\sum_{a\geq 0}\binom{n}{a}u^a\left(\frac{1-\left(1-\sum_{i\geq 1} w_{i+1}p^i\right)^k}{\left(1-\sum_{i\geq 1} w_{i+1}p^i\right)^k}\right)^{n-a} \\
=& \frac{1}{n}[p^{n-1}]\binom{n}{\ell}\left(1-\left(1-\sum_{i\geq 1} w_{i+1}p^i\right)^k\right)^{n-\ell}\left(1-\sum_{i\geq 1} w_{i+1}p^i\right)^{-k(n-\ell)} \\
=& \frac{1}{n}[p^{n-1}]\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}(-1)^a\left(1-\sum_{i\geq 1} w_{i+1}p^i\right)^{ak-k(n-\ell)} \\
=& \frac{1}{n}[p^{n-1}]\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}(-1)^a\sum_{b\geq 0}\binom{-k(n-\ell-a)}{b}\left(-\sum_{i\geq 1} w_{i+1}p^i\right)^b \\
=& \frac{1}{n}[p^{n-1}]\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}(-1)^a\sum_{b\geq 0}\binom{k(n-\ell-a)+b-1}{b}\left(\sum_{i\geq 1} w_{i+1}p^i\right)^b \\
=& \frac{1}{n}[p^{n-1}]\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}(-1)^a\sum_{b\geq 0}\binom{k(n-\ell-a)+b-1}{b} \\
& \sum_{n_2+n_3+\cdots=b}\frac{b!p^{n_2+2n_3+\cdots}w_2^{n_2}w_3^{n_3}\cdots}{n_2!n_3!\cdots} \\
=& \frac{1}{n}\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}(-1)^a\sum_{b\geq 0}\binom{k(n-\ell-a)+b-1}{b}\sum_{\substack{n_2+n_3+\cdots=b\\ n_2+2n_3+\cdots=n-1}}\frac{b!}{n_2!n_3!\cdots} \\
=& \frac{1}{n}\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}(-1)^a\sum_{b\geq 0}\binom{k(n-\ell-a)+b-1}{b}\binom{n-2}{b-1}.
\end{aligned}$$

The last equality follows by Lemma 2.3. □

Equation (15) can also be written as

$$\frac{1}{n}\binom{n}{\ell}\binom{n-2}{b-1}\sum_{a\geq 0}\binom{\ell-n+a-1}{a}\binom{k(n-\ell-a)+b-1}{b}.$$

Setting $k=1$ and summing over all values of a (making use of Identity 3.4), we obtain

$$\frac{1}{n}\binom{n}{\ell}\binom{b-1}{n-\ell-1}\binom{n-2}{b-1}$$

which is the number of plane Husimi graphs on n vertices with ℓ leaves and b blocks initially obtained by Okoth in [18].

Corollary 4.3 ([8]). *There are*

$$\frac{1}{n}\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}\binom{(k+1)n-k\ell-ka-2}{n-1}(-1)^a$$

weakly labelled k-plane trees on n vertices with ℓ leaves.

Proof. Set $b=n-1$ in (14). □

5 Forests

In this section, we count forests of weakly labelled k-plane Husimi graphs with a given number of components and blocks.

Theorem 5.1. *The number of forests of weakly labelled k-plane Husimi graphs on n vertices with r components and b blocks such that the roots of the components are labelled $1, 2, \ldots, r$ from left to right is*

$$\frac{r}{n}\binom{kn+b-1}{b}\binom{n-r-1}{b-1}. \tag{16}$$

Proof. Let $P(x)=P$ be the generating function for weakly labelled k-plane Husimi graphs with x marking vertices. Let a block of size i be marked by w_{i+1}. So,

$$P=\frac{x}{\left(1-\sum_{i\geq 1}w_{i+1}P^i\right)^k}.$$

We extract the coefficient of x^n in P^r by means of Lagrange inversion formula (Theorem 1.1):

$$\begin{aligned}
[x^n]P^r &= \frac{r}{n}[p^{n-r}]\left(1-\sum_{i\geq 1}w_{i+1}p^i\right)^{-kn}\\
&= \frac{r}{n}[p^{n-r}]\sum_{b\geq 0}\binom{-kn}{b}\left(-\sum_{i\geq 1}w_{i+1}p^i\right)^b\\
&= \frac{r}{n}[p^{n-r}]\sum_{b\geq 0}\binom{kn+b-1}{b}\left(\sum_{i\geq 1}w_{i+1}p^i\right)^b\\
&= \frac{r}{n}[p^{n-r}]\sum_{b\geq 0}\binom{kn+b-1}{b}\sum_{n_2+n_3+\cdots=b}\frac{b!\,p^{n_2+2n_3+\cdots}w_2^{n_2}w_3^{n_3}\cdots}{n_2!n_3!\cdots}\\
&= \frac{r}{n}\sum_{b\geq 0}\binom{kn+b-1}{b}\sum_{\substack{n_2+n_3+\cdots=b\\ n_2+2n_3+\cdots=n-r}}\frac{b!}{n_2!n_3!\cdots}\\
&= \frac{r}{n}\sum_{b\geq 0}\binom{kn+b-1}{b}\binom{n-r-1}{b-1}.
\end{aligned}$$

The roots of the components are then labelled $1, 2, \ldots, r$ to avoid redundancies. This completes the proof. □

Setting $r = 1$ in (16), we obtain the number of weakly labelled k-plane Husimi graphs on n vertices with b blocks. Upon setting $k = 1$ in (16), we obtain a formula for the number of forests of plane Husimi graphs on n vertices with r components and b blocks.

6 Outdegree sequence

In this section, we are interested in the number of weakly labelled tree-like structures with a given outdegree sequence.

Theorem 6.1. *The number of weakly labelled k-plane Husimi graphs on n vertices and b blocks such that there are r_j vertices with outdegree j where $j = 0, 1, \ldots, b$ is*

$$\frac{1}{n}\binom{n}{r_0, r_1, \ldots, r_b}\binom{n-2}{b-1}\binom{k}{k-1}^{r_1}\binom{k+1}{k-1}^{r_2}\cdots\binom{k+b-1}{k-1}^{r_b}. \tag{17}$$

Proof. There are

$$\frac{1}{n}\binom{n}{r_0, r_1, \ldots, r_b}\binom{n-2}{b-1}$$

plane Husimi graphs on n vertices with outdegree sequence $(r_0, r_1, \ldots, r_b)$ (see [20]). Moreover, there are $\binom{k+j-1}{k-1}$ ways to assign labels to j block children of a vertex. The result thus follows by product rule of counting. □

The following identity is obtained by summing over all values of r_i in (17) and relating it to (3).

Identity 6.2. *Let $r_0, r_1, \ldots, r_b$ be non-negative integers. Also let k and n be positive integers. Then*

$$\sum_{r_0+r_1+\cdots+r_b=n}\binom{n}{r_0, r_1, \ldots, r_b}\binom{k}{k-1}^{r_1}\binom{k+1}{k-1}^{r_2}\cdots\binom{k+b-1}{k-1}^{r_b} = \binom{kn+b-1}{b}.$$

We obtain the following corollary upon setting $b = n - 1$ in (17).

Corollary 6.3 ([8]). *The number of weakly labelled k-plane trees on n vertices such that there are r_j vertices with outdegree j where $j = 0, 1, \ldots, n-1$ is*

$$\frac{1}{n}\binom{n}{r_0, r_1, \ldots, r_{n-1}}\binom{k}{k-1}^{r_1}\binom{k+1}{k-1}^{r_2}\cdots\binom{n+k-2}{k-1}^{r_{n-1}}.$$

7 Conclusion

In this paper, the set of weakly labelled k-plane Husimi graphs are introduced and enumerated by occurrences of blocks of certain labels, number of blocks of certain block sizes, root degree and label of the eldest/youngest block child of the root, leaves, forests of weakly labelled k-plane Husimi graphs with given number of components and blocks, and outdegree sequence. The counting formulas are obtained by symbolic method and making use of Lagrange-Bürmann inversion. This work extends the previous works of Okoth [18], Onyango, Okoth and Kasyoki [20] and Kariuki and Okoth [8]. In the last paper stated, weakly labelled k-plane trees were introduced and enumerated using various parameters. Other than using symbolic methods to obtain the results in the aforesaid paper, the authors also constructed bijections between the set of these structures and other eight different combinatorial structures, i.e., k-ary Husimi graphs (studied in [20]), Dyck paths, plane trees with a given number of leaves, k-plane trees (initially introduced by Gu, Prodinger and Wagner in [4] and also studied in [14, 16, 19] among other papers), k-binary trees, S-Motzkin paths, T-Motzkin paths and ways to connect points in a line using arcs but following a certain rule. It would be interesting to investigate bijections of weakly labelled k-plane Husimi graphs.

Finding

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Data is contained within the article.

Conflicts of Interests

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