Journal of Discrete Mathematics and Its Applications 10 (3) (2025) 233–244

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Shahid Rajaee Teacher Training University

Research Paper

Generous Roman domination stability in graphs

Seyed Mahmoud Sheikholeslami[1,*], Mustapha Chellali[2], Mariyeh Kor[1]

[1]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I. R. Iran

[2]LAMDA-RO Laboratory, Department of Mathematics, University of Blida B. P. 270, Blida, Algeria

Academic Editor: Nader Jafari Rad

Abstract. Let $G=(V,E)$ be a simple graph and f a function defined from V to $\{0,1,2,3\}$. A vertex u with $f(u)=0$ is called an undefended vertex with respect to f if it is not adjacent to a vertex v with $f(v)\geq 2$. The function f is called a generous Roman dominating function (GRD-function) if for every vertex with $f(u)=0$ there exists at least a vertex v with $f(v)\geq 2$ adjacent to u such that the function $f':V\rightarrow\{0,1,2,3\}$, defined by $f'(u)=\alpha$, $f'(v)=f(v)-\alpha$ where $\alpha\in\{1,2\}$, and $f'(w)=f(w)$ if $w\in V-\{u,v\}$ has no undefended vertex. The weight of a GRD-function f is the sum of its function values over all vertices, and the minimum weight of a GRD-function on G is the generous Roman domination number $\gamma_{gR}(G)$. The γ_{gR}-stability $\mathrm{st}_{\gamma_{gR}}(G)$ (resp. γ_{gR}^{-}-stability $\mathrm{st}^{-}_{\gamma_{gR}}(G)$, γ_{gR}^{+}-stability $\mathrm{st}^{+}_{\gamma_{gR}}(G)$) of G is defined as the order of the smallest set of vertices whose removal changes (resp. decreases, increases) the generous Roman domination number. In this paper, we first determine the exact values of γ_{gR}-stability for some special classes of graphs, and then we present some bounds on $\mathrm{st}_{\gamma_{gR}}(G)$. We also characterize graphs with large $\mathrm{st}_{\gamma_{gR}}(G)$. Moreover, we show that if T is a nontrivial tree, then $\mathrm{st}_{\gamma_{gR}}(T)\leq 2$, and if further T has maximum degree $\Delta\geq 3$, then $\mathrm{st}^{-}_{\gamma_{gR}}(T)\leq\Delta-1$.

Keywords. generous Roman domination, generous Roman.
Mathematics Subject Classification (2020): 05C69.

*Corresponding author (Email address: s.m.sheikholeslami@azaruniv.ac.ir).
Received 06 March 2025; Revised 25 April 2025; Accepted 25 April 2025
First Publish Date: 01 September 2025

1 Introduction

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. Moreover, the *degree* $\deg_G(v)$ of v is $|N_G(v)|$. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. A vertex adjacent to two or more leaves is called a *strong support vertex*. If $A \subseteq V(G)$ and f is a function from $V(G)$ into some set of numbers, then $f(A) = \sum_{x \in A} f(x)$, and the sum $f(V(G))$ is called the *weight* $\omega(f)$ of f.

As usual a *path*, *cycle* and *complete graph* on n vertices are denoted by P_n, C_n and K_n. A *tree* is an acyclic connected graph. A star is the graph $K_{1,m}$, with $m \geq 1$, where the vertex of degree m of the star is called the center. A *double star* $DS_{r,s}$ is a tree obtained from two disjoint stars $K_{1,r}$ and $K_{1,s}$ by adding an edge joining their centers. The *join* $G \vee H$ of two graphs G and H is a graph formed from disjoint copies of G and H by connecting each vertex of G to each vertex of H.

Inspired by the strategies for defending the Roman Empire presented in ReVelle and Rosing [13] and Stewart [17], Cockayne et al. [10] introduced in 2004 the concept of Roman domination. But since its introduction, Roman domination has been intensively studied which led to the emergence of several variants. There are currently over 250 papers published on topics related to this concept. For more details we refer the reader to the book chapters [4,5] and surveys [6–9].

In 2024, Benatallah, Blidia and Ouldrabah [3] introduced a new variant which they called generous Roman domination defined as follows. Let f be a function defined from $V(G)$ to $\{0,1,2,3\}$. A vertex u is said to be *undefended* with respect to f if $f(u) = 0$ and u has no neighbor v with $f(v) \geq 2$. The function f is a *generous Roman dominating function* (GRD-function) if for every vertex with $f(u) = 0$ there exists at least a vertex v with $f(v) \geq 2$ adjacent to u such that the function $g : V \to \{0,1,2,3\}$ defined by $g(u) = \alpha$, $g(v) = f(v) - \alpha$ where $\alpha \in \{1,2\}$, and $g(w) = f(w)$ if $w \in V - \{u,v\}$ has no undefended vertex. The weight of a GRD-function f is the value $f(V) = \sum_{u \in V} f(u)$, and the minimum weight of a GRD-function on a graph G is the *generous Roman domination number,* abbreviated GRD-number, denoted $\gamma_{gR}(G)$. For any GRD-function f of G, let $V_i = \{v \in V \mid f(v_i) = i\}$, where $i \in \{0,1,2,3\}$. Since these four sets determine f, we can write $f = (V_0, V_1, V_2, V_3)$. Also, a $\gamma_{gR}(G)$-function is a GRD-function of G with weight $\gamma_{gR}(G)$.

In this paper, we are interested in studying the behavior of the GRD-number with respect to the deletion of a set of vertices. We therefore define the generous Roman domination stability (GRD-stability, or just γ_{gR}-stability) of a graph G as being the minimum order of a set of vertices whose removal changes the generous Roman domination number of G. On the basis of this definition, we can also define the γ_{gR}^--stability of G (resp. the γ_{gR}^+-stability) to be the minimum order of a set of vertices whose removal decreases (resp. increases) the GRD-number of G. By following the standard notations, let $\mathrm{st}_{\gamma_{gR}}(G)$, $\mathrm{st}^-_{\gamma_{gR}}(G)$ and $\mathrm{st}^+_{\gamma_{gR}}(G)$ denote the γ_{gR}-stability, γ_{gR}^--stability and γ_{gR}^+-stability, respectively. Clearly,

$\mathrm{st}_{\gamma_{gR}}(G) = \min\{\mathrm{st}^-_{\gamma_{gR}}(G), \mathrm{st}^+_{\gamma_{gR}}(G)\}$ holds for every graph G. Furthermore, it is worth noting that it is possible that the removal of any set of vertices from a graph G does not increase $\gamma_{gR}(G)$, and for such cases, we consider that $\mathrm{st}^+_{\gamma_{gR}}(G) = \infty$. The concept of stability was first studied in 1983 by Bauer et al. [2] for the domination number, and was subsequently considered for other domination parameters, including the domination number [11], the Roman domination number [12], the double Roman domination number [18], the outer independent double Roman domination number [15], the restrained domination number [1] and very recently the independent double Roman domination number [14].

In this paper, we first determine the exact values on the γ_{gR}-stability of some special classes of graphs, including paths, cycles and complete graphs. Then we present some bounds on $\mathrm{st}_{\gamma_{gR}}(G)$, and provide a characterization of all graphs with large $\mathrm{st}_{\gamma_{gR}}(G)$. Moreover, for the class of trees, we show that if T is a nontrivial tree, $\mathrm{st}_{\gamma_{gR}}(T) \leq 2$, and if further T has maximum degree $\Delta \geq 3$, then $\mathrm{st}^-_{\gamma_{gR}}(T) \leq \Delta - 1$.

2 Exact values

In this section, we determine the GRD-stability for some classes of graphs. The following two results established in [3] will be useful.

Proposition 2.1 ([3]). *For $n \geq 1$, $\gamma_{gR}(P_n) = \lceil \frac{6n}{7} \rceil$.*

Proposition 2.2 ([3]). *For $n \geq 4$, $\gamma_{gR}(C_n) = \lceil \frac{6n}{7} \rceil$.*

We first determine the γ^-_{gR}-stability for paths.

Proposition 2.3. *For $n \geq 2$,* $\mathrm{st}^-_{\gamma_{gR}}(P_n) = \begin{cases} 2 \text{ if } n \equiv 0 \pmod 7 \\ 1 \text{ otherwise.} \end{cases}$

Proof. The result is trivial when $n \in \{2,3\}$, and so we assume that $n \geq 4$. Let $P_n = v_1 v_2 \cdots v_n$. If $n \not\equiv 0 \pmod 7$, then it follows from Proposition 2.1 that $\gamma_{gR}(P_n - v_n) = \gamma_{gR}(P_{n-1}) = \left\lceil \frac{6(n-1)}{7} \right\rceil < \gamma_{gR}(P_n)$, and thus $\mathrm{st}^-_{\gamma_{gR}}(P_n) = 1$. Hence assume that $n \equiv 0 \pmod 7$. We shall first see that $\mathrm{st}^-_{\gamma_{gR}}(P_n) \geq 2$. Consider a vertex x of P_n, and observe that either $P_n - x = P_{n-1}$ or $P_n - x$ consists of two disjoint paths P_{n_1} and P_{n_2} such that $n_1 + n_2 = n - 1$. In the former case, since $n - 1 \equiv 6 \pmod 7$, we have that $\gamma_{gR}(P_{n-1}) = \lceil \frac{6n}{7} \rceil$. In the latter case we have, without loss of generality, $n_1, n_2 \equiv 3 \pmod 7$, or $n_1 \equiv 0 \pmod 7$ and $n_2 \equiv 6 \pmod 7$, or $n_1 \equiv 1 \pmod 7$ and $n_2 \equiv 5 \pmod 7$, or $n_1 \equiv 2 \pmod 7$ and $n_2 \equiv 4 \pmod 7$. In all four situations, we have through Proposition 2.1, $\gamma_{gR}(P_{n_1}) + \gamma_{gR}(P_{n_2}) = \left\lceil \frac{6n_1}{7} \right\rceil + \left\lceil \frac{6n_2}{7} \right\rceil = \lceil \frac{6n}{7} \rceil$. Hence, we conclude that $st^-_{\gamma_{gR}}(P_n) \geq 2$. Now, by Proposition 2.1, we have that $\gamma_{gR}(P_n - \{v_1, v_2\}) = \left\lceil \frac{6(n-2)}{7} \right\rceil < \gamma_{gR}(P_n)$, leading to $\mathrm{st}^-_{\gamma_{gR}}(P_n) \leq 2$. Therefore, $\mathrm{st}^-_{\gamma_{gR}}(P_n) = 2$. □

Observation 2.4. *For a path $P_n = x_1 x_2 \ldots x_n$ with $n \geq 7$, there exists a vertex $x \in V(P_n)$ such that $\gamma_{gR}(P_n) = \gamma_{gR}(P_n - x)$.*

Proof. If $n = 7$, then by Proposition 2.1, $\gamma_{gR}(P_7) = \gamma_{gR}(P_6) = \gamma_{gR}(P_n - x_7)$. Hence let $n \geq 8$. Then $P_n - x_7$ consists of two disjoint paths P_6 and P_{n-7} and again by Proposition 2.1, $\gamma_{gR}(P_n) = \lceil \frac{6n}{7} \rceil = 6 + \lceil \frac{6(n-7)}{7} \rceil = \gamma_{gR}(P_6) + \gamma_{gR}(P_{n-7})$. □

Proposition 2.5. *For $n \geq 2$, $\mathrm{st}^+_{\gamma_{gR}}(P_n) = \infty$.*

Proof. Assume for a contradiction that there exists an order n' such that $\mathrm{st}^+_{\gamma_{gR}}(P_{n'})$ is finite. In this case, among all set of vertices of cardinality $\mathrm{st}^+_{\gamma_{gR}}(P_{n'})$ whose deletion increases $\gamma_{gR}(P_{n'})$, let S be one which provides the largest number of components in the subgraph induced by $V(P_{n'}) - S$. Let $P_{n_1}, P_{n_2}, \dots P_{n_k}$ denote such components of $P_{n'} - S$. By Observation 2.4 and the highest of k, we have $n_i \leq 6$ for each $i \in \{1, \dots, k\}$. Now, if there are two consecutive vertices x_i, x_{i+1} in S, then we may assume that x_i is adjacent to some component P_{n_j}, say $j = 1$. Clearly, by Proposition 2.1, the inequality $\gamma_{gR}(P_{n'} - S) \leq \gamma_{gR}(P_{n'} - (S - \{x_i\}))$ holds, and consequently, this leads to $\gamma_{gR}(P_{n'}) < \gamma_{gR}(P_{n'} - S) \leq \gamma_{gR}(P_{n'} - (S - \{x_i\}))$. Therefore, the removal of $S - \{x_i\}$ also increases $\gamma_{gR}(P_{n'})$, thereby making $S - \{x_i\}$ smaller than S, a contradiction. Hence, S cannot contain consecutive vertices. A similar argument also shows that $x_1, x_n \notin S$. It follows that $|S| = k - 1$. Now, let t_i be the number of components of $P_{n'} - S$ of order i such that $i \in \{1,2,3,4,5,6\}$. Then $k - 1 = (\sum_{i=1}^{6} t_i) - 1$, $n' = \sum_{i=1}^{6} i t_i + (k-1)$, and so

$$\begin{aligned}
\gamma_{gR}(P_{n'} - S) &= \sum_{i=1}^{6} i t_i = \sum_{i=1}^{6} t_i \left\lceil \frac{6i}{7} \right\rceil \\
&= \sum_{i=1}^{6} \frac{6it_i + it_i}{7} = \sum_{i=1}^{6} \frac{6it_i + (k-1) - (k-1) + it_i}{7} \\
&= \frac{6n'}{7} - \sum_{i=1}^{6} \frac{(6-i)t_i - 6}{7}.
\end{aligned}$$

By a simple calculation one can conclude that $\gamma_{gR}(P_{n'} - S) \leq \lceil \frac{6n'}{7} \rceil$ leading to a contradiction. Therefore, $\mathrm{st}^+_{\gamma_{gR}}(P_n) = \infty$. □

According to Propositions 2.3 and 2.5, we immediately derive the following result.

Corollary 2.6. *For $n \geq 2$, $\mathrm{st}_{\gamma_{gR}}(P_n) = \mathrm{st}^-_{\gamma_{gR}}(P_n)$*

In the following, we consider cycles where we shall determine the GRD-stability.

Proposition 2.7. *For $n \geq 4$, $\mathrm{st}^-_{\gamma_{gR}}(C_n) = \begin{cases} 2 \text{ if } n \equiv 0 \pmod 7 \\ 1 \text{ otherwise.} \end{cases}$*

Proof. The result is trivial when $n \leq 6$, and so we assume that $n \geq 7$. Let $C_n = v_1 v_2 \dots v_n v_1$. If $n \not\equiv 0 \pmod 7$, then by Propositions 2.1 and 2.2, we have for any vertex v of C_n, $\gamma_{gR}(C_n - v) = \gamma_{gR}(P_{n-1}) = \left\lceil \frac{6(n-1)}{7} \right\rceil < \gamma_{gR}(C_n)$. Hence, in this case, $\mathrm{st}^-_{\gamma_{dR}}(C_n) = 1$.

Now, suppose that $n \equiv 0 \pmod 7$, and observe that $n-1 \equiv 6 \pmod 7$. It follows from Proposition 2.2 that $\gamma_{gR}(C_n - v_n) = \gamma_{gR}(P_{n-1}) = \left\lceil \frac{6(n-1)}{7} \right\rceil = \gamma_{gR}(C_n)$, leading that $\mathrm{st}^-_{\gamma_{gR}}(C_n) \geq 2$. Moreover, by considering the set $S = \{v_1, v_n\}$, then deleting S will provide a path of order $n-2$, and by Proposition 2.1, $\gamma_{gR}(C_n - S) = \gamma_{gR}(P_{n-2}) \leq \left\lceil \frac{6(n-2)}{7} \right\rceil < \gamma_{gR}(C_n)$. Therefore, $\mathrm{st}^-_{\gamma_{gR}}(C_n) = 2$. □

Proposition 2.8. *For $n \geq 4$, $\mathrm{st}^+_{\gamma_{gR}}(C_n) = \infty$.*

Proof. Assume for a contradiction, there exists an order n' such that $\mathrm{st}^+_{\gamma_{gR}}(C_{n'})$ is finite. Among all set of vertices of cardinality $\mathrm{st}^+_{\gamma_{gR}}(C_{n'})$ whose deletion increases $\gamma_{gR}(C_{n'})$, let S be one which provides the largest number of components in the subgraph induced by $V(C_{n'}) - S$. Let $P_{n_1}, P_{n_2}, \ldots P_{n_k}$ denote the components of $C_{n'} - S$. As in the proof of Proposition 2.5 we may assume that $n_i \leq 6$ for each $i \in \{1, \ldots, k\}$ and that S does not contain two consecutive vertices. Thus, $|S| = k$. Let t_i be the number of components of $C'_n - S$ of order i for $i \in \{1,2,3,4,5,6\}$. Then $k = (\sum_{i=1}^{6} t_i)$, $n' = \sum_{i=1}^{6} it_i + k$, and thus

$$\begin{aligned}
\gamma_{gR}(C_{n'} - S) &= \sum_{i=1}^{6} it_i = \sum_{i=1}^{6} t_i \left\lceil \frac{6i}{7} \right\rceil \\
&= \sum_{i=1}^{6} \frac{6it_i + it_i}{7} = \sum_{i=1}^{6} \frac{6it_i + k - k + it_i}{7} \\
&= \frac{6n'}{7} - \sum_{i=1}^{6} \frac{(6-i)t_i}{7} \leq \lceil \frac{6n'}{7} \rceil,
\end{aligned}$$

a contradiction. Thus $\mathrm{st}^+_{\gamma_{gR}}(C_n) = \infty$. □

According to Propositions 2.7 and 2.8, we immediately derive the following result.

Corollary 2.9. *For $n \geq 4$, $\mathrm{st}_{\gamma_{gR}}(C_n) = \mathrm{st}^-_{\gamma_{gR}}(C_n)$*

We close this section by providing the γ_{gR}-stability of three specific graphs, namely complete graphs, stars and double stars. For this purpose, we need the following Observations.

Observation 2.10 ([3]). *For $n \geq 2$, $\gamma_{gR}(K_n) = 2$ and for $n \geq 3$, $\gamma_{gR}(K_{1,n-1}) = 3$.*

Observation 2.11. *For $1 \leq r \leq t$,*

$$\gamma_{gR}(S_{r,t}) = \begin{cases} 4 \text{ if } r = 1 \\ 5 \text{ if } r = 2 \\ 6 \text{ if } r \geq 3 \end{cases}$$

From the previous observations, one can easily obtain the following.

Corollary 2.12. $\mathrm{st}_{\gamma_{gR}}(K_n) = \mathrm{st}^-_{\gamma_{gR}}(K_n) = n-1$ *and* $\mathrm{st}^+_{\gamma_{gR}}(K_n) = \infty$.

Corollary 2.13. *For $n \geq 3$, $\mathrm{st}^-_{\gamma_{gR}}(K_{1,n-1}) = n-2$, $\mathrm{st}^+_{\gamma_{gR}}(K_{1,n-1}) = 1$ for $n \geq 5$ and $\mathrm{st}^+_{\gamma_{gR}}(K_{1,n-1}) = \infty$ for $n \in \{2,3,4\}$.*

Corollary 2.14. *For $1 \leq r \leq t$, $\mathrm{st}^-_{\gamma_{gR}}(S_{r,t}) = 1$ for $r \in \{1,2,3\}$ and $\mathrm{st}^-_{\gamma_{gR}}(S_{r,t}) = r-2$ for $r \geq 4$. Furthermore, $\mathrm{st}^+_{\gamma_{gR}}(S_{r,t}) = 1$ for $t \geq 4$.*

3 Bounds and graphs with large GRD-stability

In this section, we present some bounds on the GRD-stability and we characterize graphs with large GRD-stability. Since for any graph of order $n \geq 2$, $\gamma_{gR}(G) \geq 2$ with equality if and only if $G = K_n$, the first bound is obtained.

Proposition 3.1. *Let G be a connected graph of order $n \geq 2$. Then $\mathrm{st}_{\gamma_{gR}}(G) \leq n-1$ with equality if and only if $G = K_n$.*

Proof. If G is a nontrivial complete graph K_n, then $\gamma_{gR}(G) = 2$ and clearly $\mathrm{st}_{\gamma_{gR}}(G) = n-1$. Now, assume that G is not a complete graph K_n. Then $n \geq 3$ and thus $\gamma_{gR}(G) \geq 3$. Since the GRD-number of any graph with two vertices is 2, we deduce that $\mathrm{st}_{\gamma_{gR}}(G) \leq n-2$, as desired. □

We now give a characterization of connected graphs with $\gamma_{gR}(G) = 3$ which will be useful in what follows.

Proposition 3.2. *Let $G \neq K_n$ be a connected graph of order $n \geq 3$. Then $\gamma_{gR}(G) = 3$ if and only if $\Delta(G) = n-1$.*

Proof. Let $G \neq K_n$ be a connected graph of order $n \geq 3$. Clearly $\gamma_{gR}(G) \geq 3$. If $\Delta(G) = n-1$, then assigning 3 to a vertex of degree $n-1$ and 0 to any other vertex, provides a GRD-function of G of weight 3 and consequently $\gamma_{gR}(G) = 3$.

Conversely, assume that $\gamma_{gR}(G) = 3$ and let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{gR}(G)$-function such that $|V_3|$ is as large as possible. If $n = 3$, then it follows from the connectedness of G that $\Delta(G) = 2 = n-1$. Hence let $n \geq 4$. Since $\gamma_{gR}(G) = |V_1| + 2|V_2| + 3|V_3|$, one of the two possibilities holds either (i) $|V_3| = 1$ and $|V_1| = |V_2| = 0$ or (ii) $|V_3| = 0$ and $|V_1| = |V_2| = 1$. If $|V_3| = 1$ and $|V_1| = |V_2| = 0$, then the vertex in V_3 is adjacent to all vertices of G leading to $\Delta(G) = n-1$. Now, assume that $|V_1| = |V_2| = 1$ and $|V_3| = 0$. Let $V_1 = \{x\}$ and $V_2 = \{y\}$. Then every vertex in $V(G) - \{x,y\}$ must be adjacent to y and it follows from $n \geq 4$ and the definition of GRD-function that the graph $G - x$ is complete. On the other hand, since G is connected x must be adjacent to some vertex z in $G - x$, implying that $\deg(z) = n-1$ and consequently $\Delta(G) = n-1$. □

Proposition 3.3. *Let G be a connected graph of order $n \geq 3$ such that $\gamma_{gR}(G) = 3$. Then $\mathrm{st}_{\gamma_{gR}}(G) \leq \lfloor \frac{n}{2} \rfloor$, with equality if and only if $G = K_{1,3}$ or $G = K_{\lfloor \frac{n}{2} \rfloor} \vee \overline{K_{\lceil \frac{n}{2} \rceil}}$, for an odd n.*

Proof. By Proposition 3.2, $\Delta(G) = n-1$. Since the result can be easily checked for $n \in \{3,4\}$, we assume that $n \geq 5$. Let U be the set of vertices of G with maximum degree $\Delta(G)$. It follows from $\gamma_{gR}(G) = 3$ that $|U| \leq n-2$. Moreover, if $|U| \geq \lfloor \frac{n}{2} \rfloor$, then for any vertex $x \in V(G) - U$, the graph $G[U \cup \{x\}]$ is a complete graph and so $\gamma_{gR}(G[U \cup \{x\}]) = 2$, thus $\mathrm{st}_{\gamma_{gR}}(G) \leq n - |U| - 1 \leq n - \lfloor \frac{n}{2} \rfloor - 1 \leq \lfloor \frac{n}{2} \rfloor$. If further, $\mathrm{st}_{\gamma_{gR}}(G) = \lfloor \frac{n}{2} \rfloor$, then we have equality throughout the previous inequality chain. In particular, $|U| = \lfloor \frac{n}{2} \rfloor$ and $n - \lfloor \frac{n}{2} \rfloor - 1 = \lfloor \frac{n}{2} \rfloor$, which implies that n is odd. Now, if there is an edge xy in $G - U$, then $G[U \cup \{x,y\}]$ is a complete graph and so $\gamma_{gR}(G[U \cup \{x,y\}]) = 2$ leading to $\mathrm{st}_{\gamma_{gR}}(G) < \lfloor \frac{n}{2} \rfloor$, a contradiction. Thus $G - U$ is edgeless and thus $G = K_{\lfloor \frac{n}{2} \rfloor} \vee \overline{K_{\lceil \frac{n}{2} \rceil}}$ where n is odd. Henceforth, we can now assume that $|U| < \lfloor \frac{n}{2} \rfloor$. Then $G - U$ is a graph of order at least 3 with $\Delta(G) \leq n - |U| - 2$. By Proposition 3.2, it follows that $\gamma_{gR}(G - U) \neq 3$ and thus $\mathrm{st}_{\gamma_{gR}}(G) \leq |U| < \lfloor \frac{n}{2} \rfloor$ as desired. This completes the proof. □

Proposition 3.4. *If G is a graph of order $n \geq 2$, then $\mathrm{st}_{\gamma_{gR}}(G) \leq \delta(G) + 1$.*

Proof. Let v be a vertex of G with $\deg_G(v) = \delta(G)$, and let $G' = G - N(v)$ and $G'' = G - N[v]$. If $\delta(G) = 0$, then v is isolated, and thus the result is valid. Hence we assume that $\delta(G) \geq 1$. If $\gamma_{gR}(G') \neq \gamma_{gR}(G)$, then $\mathrm{st}_{\gamma_{gR}}(G) \leq \delta(G) < \delta(G) + 1$. Hence assume that $\gamma_{gR}(G') = \gamma_{gR}(G)$, and let f be a $\gamma_{gR}(G')$-function. Since v is isolated in G', $f(v) = 1$, and thus $\gamma_{gR}(G') = \gamma_{gR}(G'') + 1$. t follows that $\gamma_{gR}(G'') = \gamma_{gR}(G) - 1$ and therefore $\mathrm{st}_{\gamma_{gR}}(G) \leq \delta(G) + 1$. □

Proposition 3.5. *Let G be a graph with $\gamma_{gR}(G) \geq 4$, then $\mathrm{st}_{\gamma_{gR}}(G) \leq n - \Delta(G) - 1$.*

Proof. It follows from $\gamma_{gR}(G) \geq 4$ that $\Delta(G) \leq n - 2$ and also $\Delta(G) \geq 2$. Since the result is valid when G has an isolated vertex, we can assume that $\delta(G) \geq 1$. Let v be a vertex of G with $\deg_G(v) = \Delta(G)$. Restricted to the subgraph induced by v and its neighbors, we deduce from Proposition 3.2 that $\gamma_{gR}(G[N[v]]) = 3$, and thus $\gamma_{gR}(G[N[v]]) < \gamma_{gR}(G)$. Therefore removing all vertices not in $N[v]$ changes the GRD-number of G, leading to $\mathrm{st}_{\gamma_{gR}}(G) \leq |V(G) \setminus N[v]| = n - \Delta(G) - 1$. □

Combining Propositions 3.4 and 3.5, the following result is immediate.

Corollary 3.6. *Let G be a graph with $\gamma_{gR}(G) \geq 4$, then $\mathrm{st}_{\gamma_{gR}}(G) \leq \min\{\delta(G) + 1, n - \Delta(G) - 1\}$.*

In what follows, we provide a characterization of connected graphs with $\mathrm{st}_{\gamma_{gR}}(G) \in \{n - 2, n - 3\}$.

Theorem 3.7. *If $G \neq K_n$ is a connected graph of order $n \geq 3$, then $\mathrm{st}_{\gamma_{gR}}(G) = n - 2$ with equality if and only if $G \in \{P_3, K_{1,3}\}$.*

Proof. If $G \in \{P_3, K_{1,3}\}$, then it is easy to verify that $\mathrm{st}_{\gamma_{gR}}(G) = n - 2$. To prove the necessity, let G be a connected graph with $\mathrm{st}_{\gamma_{gR}}(G) = n - 2$. Note that $\Delta(G) \geq 2$, since G is a connected graph having at least three vertices. Now, if $\gamma_{gR}(G) \geq 4$, then by Proposition 3.4, $\Delta(G) \leq n - \mathrm{st}_{\gamma_{gR}}(G) - 1 = 1$, a contradiction. Hence, $\gamma_{gR}(G) = 3$. It follows from Proposition 3.3 that $n - 2 = \mathrm{st}_{\gamma_{gR}}(G) \leq \lfloor \frac{n}{2} \rfloor$ and thus $n \in \{3, 4\}$. Now, it is easy to see that $G \in \{P_3, K_{1,3}\}$. □

Let H denote the paw graph, i.e, H is the graph obtained from K_3 by adding a new vertex that we join to a vertex of K_3.

Theorem 3.8. *If G is a connected graph of order $n \geq 4$ such that $G \notin \{K_n, K_{1,3}\}$, then $\mathrm{st}_{\gamma_{gR}}(G) \leq n - 3$, with equality if and only if $G \in \{P_4, C_4, K_4 - e, H, K_2 \vee \overline{K_3}\}$.*

Proof. The upper bound follows from Proposition 3.1 and Theorem 3.7. Moreover, if $G \in \{P_4, C_4, K_4 - e, H, K_2 \vee \overline{K_3}\}$, then one can easily check that $\mathrm{st}_{\gamma_{gR}}(G) = n - 3$. We now prove the necessity. Assume that $\mathrm{st}_{\gamma_{gR}}(G) = n - 3$. If $\gamma_{gR}(G) \geq 4$, then it follows from Proposition 3.5 that $\Delta(G) \leq n - \mathrm{st}_{\gamma_{gR}}(G) - 1 = 2$ and thus G is a path or a cycle. By Corollaries 2.6 and 2.9, we deduce that $G \in \{P_4, C_4\}$. In the sequel, we can assume that $\gamma_{gR}(G) = 3$. First let n be even. In this case, by Proposition 3.3, $n - 3 = \mathrm{st}_{\gamma_{gR}}(G) \leq \frac{n}{2} - 1$ leading to $n = 4$. By Proposition 3.2, $\Delta(G) = n - 1 = 3$, and thus $G = H$ or $G = K_4 - e$. Now let n be odd. By Proposition 3.3, $n - 3 = \mathrm{st}_{\gamma_{gR}}(G) \leq \frac{n-1}{2}$ leading that $n = 5$. Since $\mathrm{st}_{\gamma_{gR}}(G) = n - 3 = 2 = \lfloor \frac{5}{2} \rfloor$, it follows from Proposition 3.3 that $G = K_2 \vee \overline{K_3}$, and the proof is complete. □

The next corollary is an immediate consequence of Proposition 3.1 and Theorems 3.7 and 3.8.

Corollary 3.9. *If $G \neq K_n$ is a connected graph of order $n \geq 6$, then $\mathrm{st}_{\gamma_{gR}}(G) \leq n - 4$.*

We close this section by presenting a Nordhaus-Gaddum type inequality for the sum of the GRD-stability of a graph G and the complement $\overline{G}$ of G.

Theorem 3.10. *Let G be a graph of order $n \geq 2$. Then $\mathrm{st}_{\gamma_{gR}}(G) + \mathrm{st}_{\gamma_{gR}}(\overline{G}) \leq n$ with equality if and only if $G \in \{K_n, \overline{K_n}\}$.*

Proof. If G is a complete graph, then $\overline{G}$ is an edgeless graph, and clearly $\mathrm{st}_{\gamma_{gR}}(\overline{G}) = 1$, while by Proposition 3.1 $\mathrm{st}_{\gamma_{gR}}(K_n) = n - 1$, and thus $\mathrm{st}_{\gamma_{gR}}(G) + \mathrm{st}_{\gamma_{gR}}(\overline{G}) = n$. Likewise, if $G = \overline{K_n}$, then we have $\mathrm{st}_{\gamma_{gR}}(G) + \mathrm{st}_{\gamma_{gR}}(\overline{G}) = n$. In the next, we may assume that $G \notin \{K_n, \overline{K_n}\}$. Thus $n \geq 3$, and clearly $\min\{\gamma_{gR}(G), \gamma_{gR}(\overline{G})\} \geq 3$. If $\gamma_{gR}(G) = 3$ (the case $\gamma_{gR}(\overline{G}) = 3$ is similar), then by Proposition 3.2, $\Delta(G) = n - 1$, and so $\overline{G}$ has an isolated vertex. In which case, it is clear that $\mathrm{st}_{\gamma_{gR}}(\overline{G}) = 1$. Moreover, by Proposition 3.3, $\mathrm{st}_{\gamma_{gR}}(G) \leq \lfloor \frac{n}{2} \rfloor$, and thus $\mathrm{st}_{\gamma_{gR}}(G) + \mathrm{st}_{\gamma_{gR}}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor + 1 < n$. In the following, we can assume that $\min\{\gamma_{gR}(G), \gamma_{gR}(\overline{G})\} \geq 4$. Since $\Delta(G) + \Delta(\overline{G}) \geq n - 1$, we may assume, without loss of generality, that $\Delta(G) \geq (n-1)/2$. If $\Delta(\overline{G}) \geq (n-2)/2$, then using Corollary 3.6 we have

$$\begin{aligned} \mathrm{st}_{\gamma_{gR}}(G) + \mathrm{st}_{\gamma_{gR}}(\overline{G}) &\leq (n - \Delta(G) - 1) + (n - \Delta(\overline{G}) - 1) \\ &\leq (n - \frac{n-1}{2} - 1) + (n - \frac{n-2}{2} - 1) \\ &= \frac{2n-1}{2} < n. \end{aligned}$$

Hence, we assume that $\Delta(\overline{G}) < (n-2)/2$ and so $\delta(\overline{G}) < (n-2)/2$. Applying Corollary 3.6, leads to

$$\begin{aligned} \mathrm{st}_{\gamma_{gR}}(G) + \mathrm{st}_{\gamma_{gR}}(\overline{G}) &\leq (n - \Delta(G) - 1) + (\delta(\overline{G}) + 1) \\ &\leq (n - \frac{n-1}{2} - 1) + \frac{n}{2} \\ &= \frac{2n-1}{2} < n, \end{aligned}$$

as desired. This completes the proof □

4 Trees

In this section, we determine the $\mathrm{st}_{gR}(T)$-stability and the $\mathrm{st}_{gR}(T)$-stability for trees. Note that according to Proposition 2.5, $\mathrm{st}^{+}_{gR}(T)$ cannot be bounded.

Theorem 4.1. *For every tree T of order $n \geq 2$, $\mathrm{st}_{gR}(T) \leq 2$.*

Proof. Assume for a contradiction that there exists a tree T such that $\mathrm{st}_{\gamma_{gR}}(T) > 2$. Among all such trees, we assume that T has a smallest order. By Corollaries 2.13 and 2.14, T is neither a star nor a double star and thus it has a diameter, $\mathrm{diam}(T)$, at least 4. Let $x_1x_2\ldots x_d$ be a diametral path in T. Note that x_1 and x_d are leaves. Let $x_1 = y_1, y_2, \ldots, y_t$ be the leaf neighbors of x_2, and let T_{x_2} denote the subtree of T induced by x_2 and its leaf neighbors. If $t \geq 4$, then $T - x_2$ is a forest, and clearly $\gamma_{gR}(T - x_2) = \gamma_{gR}(T - T_{x_2}) + t$. Moreover, any $\gamma_{gR}(T - T_{x_2})$-function can be extended to a GRD-function of T by assigning 3 to x_2 and 0 to $y_1, \ldots, y_t$ leading to $\gamma_{gR}(T) \leq \gamma_{gR}(T - T_{x_2}) + 3 < \gamma_{gR}(T - x_2)$, because of $t \geq 4$. Hence if $t \geq 4$, then $\mathrm{st}_{gR}(T) = 1$, a contradiction. Therefore, $t \leq 3$. Let $L(T)$ denote the set of leaves of T, and let f be a $\gamma_{gR}(T)$-function such that $f(L(T))$ is as small as possible. We consider three cases.

Case 1. $t = 3$.
By the choice of f, we must have $f(x_2) = 3$ and $f(y_1) = f(y_2) = f(y_3) = 0$. Since it is assumed that $\mathrm{st}_{\gamma_{gR}}(T) > 2$, it follows that removing x_2 does not change the GDR-number of T, and thus $\gamma_{gR}(T) = \gamma_{gR}(T - x_2) = \gamma_{gR}(T - T_{x_2}) + 3$. Moreover, since any $\gamma_{gR}(T - T_{x_2})$-function can be extended to a GRD-function of the tree $T' = T - \{y_2, y_3\}$ by assigning 2,0 to x_2, x_1 respectively. It follows that $\gamma_{gR}(T') \leq \gamma_{gR}(T - T_{x_2}) + 2 = \gamma_{gR}(T) - 1$, leading that $\mathrm{st}_{gR}(T) \leq 2$, a contradiction.

Case 2. $t = 2$.
If $f(x_2) = 3$, then reassigning the value 2 to x_2 provides a GRD-function of $T - \{y_1, y_2\}$ implying that $\mathrm{st}^{-}_{\gamma_{gR}}(T) \leq 2$, a contradiction. Hence we assume that $f(x_2) \leq 2$. It follows from the choice of f that $f(x_2) = 0$ and so $f(y_1) = f(y_2) = 1$. Then the restriction of f to $T - x_1$ is a GRD-function of $T - x_1$ with weight $\omega(f) - 1$, implying that $\mathrm{st}_{gR}(T) = 1$, a contradiction.

Case 3. $t = 1$.
Thus $\deg_T(x_2) = 2$. If $f(x_2) \geq 2$, then the function g defined on $T - \{x_2, x_1\}$ by $g(x_3) = \min\{3, f(x_3) + 1\}$ and $g(x) = f(x)$ for any other vertex, is a GRD-function of $T - \{x_1, x_2\}$ of weight less than $\omega(f)$ leading to $\mathrm{st}_{gR}(T) \leq 2$, a contradiction. Hence assume that $f(x_2) \leq 1$. Then $f(x_1) = 1$ and thus the restriction of f to $T - x_1$ is a GRD-function of $T - x_1$ with weight $\omega(f) - 1$, leading to the contradiction that $\mathrm{st}_{gR}(T) = 1$. Therefore, $\mathrm{st}_{gR}(T) \leq 2$ and this completes the proof. □

Theorem 4.2. *Let T be a tree of order $n \geq 3$ with maximum degree $\Delta \geq 3$. Then $\mathrm{st}^{-}_{\gamma_{gR}}(T) \leq \Delta - 1$. Furthermore, the bound is sharp for stars $K_{1,\Delta}$.*

Proof. If T is the star $K_{1,\Delta}$, then by Corollary 2.13, we have $\mathrm{st}^{-}_{\gamma_{gR}}(T) = \Delta - 1$. Hence we assume that T is not a star, and thus T has diameter at least 3. Let $x_1x_2\ldots x_d$ be a diametral path in G. Clearly, x_1 and x_d are leaves. Let $y_1 (= x_1), y_2, \ldots, y_t$ be the leaf neighbors of x_2, and note that $t \leq \Delta - 1$, because of $\deg_T(x_2) \leq \Delta$. Let f be a $\gamma_{gR}(T)$-function that assigns the smallest possible values to the leaves. If $t \geq 3$, then $f(x_2) = 3$ and reassigning the value 2 to x_2 provides a GRD-function of $T - \{y_1, \dots, y_t\}$ implying that $\mathrm{st}^{-}_{\gamma_{gR}}(T) \leq \Delta - 1$. Assume now that $t = 2$. If $f(x_2) = 3$, then the result follows as above. Thus we assume that $f(x_2) \leq 2$. It follows from the choice of f that $f(x_2) = 0$ and so $f(y_1) = f(y_2) = 1$. Then the restriction of f to $T - x_1$ is a GRD-function of $T - x_1$ with weight $\omega(f) - 1$ and therefore $\mathrm{st}^{-}_{gR}(T) = 1 < \Delta - 1$. Finally assume that $t = 1$. If $f(x_2) = 3$, then the result follows as above. Thus we can assume that $f(x_2) \leq 2$. If $f(x_2) \leq 1$, then by the choice of f we have $f(x_1) = 1$ and the restriction of f to $T - x_1$ is a GRD-function of $T - x_1$ with weight $\omega(f) - 1$ leading as before $\mathrm{st}^{-}_{gR}(T) = 1 < \Delta - 1$. Henceforth, we assume that $f(x_2) = 2$. Then the function g defined on $T - \{x_1, x_2\}$ by $g(x_3) = \min\{3, f(x_3) + 1\}$ and $g(x) = f(x)$ for any other vertex is a GRD-function of $T - \{x_1, x_2\}$ of weight $\omega(f) - 1$ and thus $\mathrm{st}^{-}_{gR}(T) \leq 2 \leq \Delta - 1$. □

The next result is an immediate consequence of Proposition 2.3 and Theorem 4.2.

Corollary 4.3. *Let T be a tree of order $n \geq 3$ with maximum degree Δ. Then $\mathrm{st}^{-}_{\gamma_{gR}}(T) \leq \Delta$ with equality if and only if $T = P_{7k}$ for some positive integer k.*

We conclude this section with a problem.

Open Problem 4.4. *Characterize all trees T with maximum degree at least three and $\mathrm{st}^{-}_{gR}(T) = \Delta - 1$.*

5 Conclusion

In this paper, we have studied the generous Roman domination stability. We determined exact values of the generous Roman domination stability for special classes of graphs. Additionally, we established bounds on the γ_{gR}-stability for general graphs. For trees, we proved that the γ_{gR}-stability is at most two, while the γ^{-}_{gR}-stability is bounded above by the maximum degree $\Delta - 1$ of the tree. The problem of characterization of the trees that attain this upper bound is open.

Finding

This research received no external funding.

Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

References

[1] A. A. Aghdash, N. Jafari Rad, B.V. Fasaghandisi, On the restrained domination stability in graphs, RAIRO-Oper. Res. 59 (2025) 579–586. https://doi.org/10.1051/ro/2024233

[2] D. Bauer, F. Harary, J. Nieminen, C. Suffel, Domination alternation sets in graphs, Discrete Math. 47 (1983) 153–161. https://doi.org/10.1016/0012-365X(83)90085-7

[3] M. Benatallah, M. Blidia, L. Ouldrabah, The generous Roman domination number, Trans. Comb. 13 (2024) 179–196. https://doi.org/10.22108/toc.2023.131167.1928

[4] M. Chellali, N. Jafari Rad, S. M. Sheikholeslami, L. Volkmann, Roman domination in graphs, Topics in Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg (2020) 365–409. https://doi.org/10.1007/978-3-030-51117-3

[5] M. Chellali, N. Jafari Rad, S. M. Sheikholeslami, L. Volkmann, Varieties of Roman domination, Structures of Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg (2021) 273–307. https://doi.org/10.1007/978-3-030-58892-2_10

[6] M. Chellali, N. Jafari Rad, S. M. Sheikholeslami, L. Volkmann, Varieties of Roman domination II, AKCE Int. J. Graphs Comb.17 (2020) 966–984. https://doi.org/10.1016/j.akcej.2019.12.001

[7] M. Chellali, N. Jafari Rad, S. M. Sheikholeslami, L. Volkmann, Varieties of Roman domination III, submitted.

[8] M. Chellali, N. Jafari Rad, S. M. Sheikholeslami, L. Volkmann, The Roman domatic problem in graphs and digraphs: A survey, Discuss. Math. Graph Theory 42 (2022) 861–891. https:// doi.org/ 10.7151/ dmgt.2313

[9] M. Chellali, N. Jafari Rad, S. M. Sheikholeslami, L. Volkmann, A survey on Roman domination parameters in directed graphs, J. Combin. Math. Combin. Comput. 115 (2020) 141–171. https:// combinatorialpress.com/ jcmcc-articles/ volume-115

[10] E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi, S. T. Hedetniemi, Roman domination in graphs, Discrete Math. 278 (2004) 11–22. https://doi.org/10.1016/j.disc.2003.06.004

[11] N. Jafari Rad, E. Sharifi, M. Krzywkowski, Domination stability in graphs, Discrete Math. 339 (2016) 1909–1914. https://doi.org/10.1016/j.disc.2015.12.026

[12] M. Hajian, N. Jafari Rad, On the Roman domination stable graphs, Discuss. Math. Graph Theory 37 (2017) 859–871. https://doi.org/10.7151/dmgt.1975

[13] C. S. ReVelle, K. E. Rosing, Defendens imperium romanum: a classical problem in military strategy, Amer. Math. Monthly 107 (7) (2000) 585–594. https://doi.org/10.1080/00029890.2000.12005243

[14] S. M. Sheikholeslami, M. Esmaeili, J. J. Hamja, C. L. Armada, I. S. Aniversario, Independent double Roman domination stability in graphs, Euro. J. Pure Appl. Math. 18 (2025) 5984. https:// doi.org/ 10.29020/ nybg.ejpam.v18i2.5984

[15] S. M. Sheikholeslami, M. Esmaeili, L. Volkmann, Outer independent double Roman domination stability in graphs, Ars Combin. 160 (2024) 21–29. https:// doi.org/ 10.61091/ ars-160-04

[16] S. M. Sheikholeslami, M. Chellali, M. Kor, Further results on generous Roman domination, Math. Interdisc. Res. 10(2) (2025) 231–243. https://doi.org/10.22052/mir.2025.256617.1511

[17] I. Stewart, Defend the Roman Empire!, Sci. Amer. 281(6) (1999) 136–139. https://doi.org/10.1038/scientificamerican1299-136

[18] W. Zhuang, Double Roman domination stability in graphs, Discrete Appl. Math. 371 (2025) 254–263. https://doi.org/10.1016/j.dam.2025.04.035

Citation: S. M. Sheikholeslami, M. Chellali, M. Kor, Generous Roman domination stability in graphs, J. Disc. Math. Appl. 10(3) (2025) 233–244.

https://doi.org/10.22061/jdma.2025.11827.1119