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Research Paper

The structure and parameters of a graph assigned to topological spaces

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Abstract. For a set X and a topological space (X, τ), let $\Gamma_X(\tau)$ be a graph with vertex set $\tau \setminus \{\emptyset, X\}$ in which two vertices A_1 and A_2 are adjacent just when $A_1 \cup A_2 = X$. In this paper and among some other results, we study the maximum and minimum degrees, the matching number, the chromatic number, the chromatic index, the planarity, the Wiener index and the Zagreb index of $\Gamma_X(\tau)$ and we determine their exact values in general cases or in some special topological spaces like T_1.

Keywords. Topological space, graph coloring, planarity, topological indices.
Mathematics Subject Classification (2020): 05C15, 05C10.

1 Introduction

A topological space is a set endowed with a structure, called a topology, which allows defining continuous deformation of subspaces and (more generally) all kinds of continuity. Euclidean spaces and (more generally) metric spaces are examples of topological spaces, as every distance or metric defines a topology, see [7] for more details. Formally, let X be a set and τ be a family of subsets of X. Then τ is called a topology on X if the following three properties are satisfied.

1) Both the empty set $\emptyset$ and the whole set X are elements of τ.

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2) Any union of elements of τ is an element of τ.

3) Any intersection of finitely many elements of τ is an element of τ.

If τ is a topology on X, then the pair (X, τ) is called a topological space and each member of τ is an open set. A T_1 space is a topological space in which, for every pair of distinct points, each has a neighborhood (an open set containig it) not containing the other point. There are several graphs assigned to topological spaces in litherature and vice versa, see [2], [5] and [8]. Let (X, τ) be a topological space. The graph $\Gamma_X(\tau)$ is a graph with vertex set $\tau \setminus \{\emptyset, X\}$ in which two vertices A_1 and A_2 are adjacent just when $A_1 \cup A_2 = X$. Let Γ be a finite and simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. When two vertices u and v are adjacent, we can denote it by the notation $u \sim v$. The set $N_\Gamma(u)$ denotes the (open) neighborhood of $u \in V(\Gamma)$, which means the set of all adjacent vertices to u in Γ and the degree of u is $\deg_\Gamma(u) = |N_\Gamma(u)|$, see [10]. An isolated vertex is a vertex of degree zero. For a subset of vertices $X \subseteq V$, $\Gamma[X]$ denotes the subgraph induced by X. Note that X is an independent set in Γ if and only if $\Gamma[X]$ contains no edge. A clique C in Γ is a subset of vertices of Γ such that every two distinct vertices in C are adjacent and hence, the induced subgraph of Γ on it is a complete graph. The clique number of Γ, $\omega(\Gamma)$, is the maximum size among all cliques of Γ. A matching in Γ is a set of pairwise non-incident edges of $E(\Gamma)$ and, a perfect matching is a matching in which every vertex of the graph is incident to exactly one edge of the matching. A (proper vertex) k-coloring of Γ is an assignment of k colors to the vertices of Γ in such a way that no pair of adjacent vertices receive the same color, and the chromatic number $\chi(\Gamma)$ is the minimum integer k for which a k-coloring for Γ exists. Similarly, a (proper) edge coloring of a graph is an assignment of colors to the edges such that no two incident edges received the same color and the minimum required number of colors for such coloring is called the chromatic index (or the edge chromatic number) of the graph and is denoted by $\chi'(\Gamma)$. A topological index (a molecular structure descriptor) for a graph Γ is a numerical quantity invariant under automorphisms of Γ and it does not depend on the labeling or pictorial representation of the graph. One of the oldest and most thoroughly studied distance based topological index is the Wiener index, and the first Zagreb index is the oldest and most studied vertex-degree based topological indical index. The Wiener index $W(\Gamma)$ of Γ and the first Zagreb index $Z_1(\Gamma)$ of Γ are defined as follows (see [11] and [4], respectively)

$$W(\Gamma) = \sum_{\{u,v\} \subseteq V(\Gamma)} d_\Gamma(u,v) \ , \ Z_1(\Gamma) = \sum_{u \in V(\Gamma)} \deg_\Gamma(u)^2,$$

in which $d_\Gamma(u,v)$ denotes the distance of two vertices u and v in Γ. For a review in these subjects and related subjects, one can see [1], [3], [6], [8] and [9]. In this paper and among some othe results, we study the maximum and minimum degrees, the matching number, the chromatic number, the chromatic index, the planarity, the Wiener index and the Zagreb index of $\Gamma_X(\tau)$ and we determine their exact values in general cases or in some special topological spaces like T_1.

2 Main results

Proposition 2.1. *Let X be a set and τ_1, τ_2 be two topologies on X. Then, τ_1 is finer (stronger or larger) topology than τ_2 if and only if $\Gamma_X(\tau_2)$ is an induced subgraph of $\Gamma_X(\tau_1)$.*

Proof. Assume that τ_1 is a finer topology than τ_2 and hence, $\tau_2 \subseteq \tau_1$. Thus, $\tau_2 \setminus \{\emptyset, X\} \subseteq \tau_1 \setminus \{\emptyset, X\}$ which means that the vertex set of $\Gamma_X(\tau_2)$ is a subset of the vertex set of $\Gamma_X(\tau_1)$. Let G and H be two vertices in the vertex set of $\Gamma_X(\tau_2)$ and asuume that they are adjacent in the graph $\Gamma_X(\tau_1)$. This means that $G \cup H = X$ (in τ_1). The relation $G \cup H = X$ implies that G and H are two adjacent vertices in $\Gamma_X(\tau_2)$. Thus, $\Gamma_X(\tau_2)$ is an induced subgraph of $\Gamma_X(\tau_1)$. The converse is similarly done. □

Proposition 2.2. *Let (X, τ_1) and (Y, τ_2) be two topological spaces and $f : X \to Y$ be a (continius) function such that $f^{-1}(U)$ is a non-trivial open set in X for each non-trivial open set U in Y. Then, the function $F : V(\Gamma_Y(\tau_2)) \to V(\Gamma_X(\tau_1))$ defined by $F(U) = f^{-1}(U)$ is a graph homomorphism.*

Proof. For this purpose, we must show that $F(G)$ and $F(H)$ are adjacent in $\Gamma_X(\tau_1)$ for each pair of adjacent vertices G and H in $\Gamma_Y(\tau_2)$. This is certainly satisfied, beceause when G and H are two adjacent vertices in $\Gamma_Y(\tau_2)$, then we have $G \cup H = Y$ and hence,

$$F(G) \cup F(H) = f^{-1}(G) \cup f^{-1}(H) = f^{-1}(G \cup H) = f^{-1}(Y) = X,$$

which implies that $F(G)$ and $F(H)$ are two adjacent vertices in $\Gamma_X(\tau_1)$. □

Proposition 2.3. *Let (X, τ) be a topological space and G_1, G_2 be two vertices in $\Gamma_X(\tau)$ (i.e., two non-trivial open sets) such that $G_1 \subseteq G_2$. Then, we have $\deg_{\Gamma_X(\tau)}(G_1) \leq \deg_{\Gamma_X(\tau)}(G_2)$.*

Proof. Let $H \in \tau$ be a neighbour of G_1 in the graph $\Gamma_X(\tau)$. This means that $G_1 \cup H = X$. Since $G_1 \subseteq G_2$, thus we have

$$X = G_1 \cup H \subseteq G_2 \cup H \subseteq X,$$

which implies that $G_2 \cup H = X$. Thus, H is a neighbour of G_1 in $\Gamma_X(\tau)$. This implies that $N_{\Gamma_X(\tau)}(G_1) \subseteq N_{\Gamma_X(\tau)}(G_2)$ and hence, $\deg_{\Gamma_X(\tau)}(G_1) \leq \deg_{\Gamma_X(\tau)}(G_2)$. □

Proposition 2.4. *Let (X, τ) be a T_1 topological space. Then, for each vertex G in $\Gamma_X(\tau)$ we have $\deg_{\Gamma_X(\tau)}(G) \geq 2^{|G|} - 1$. Especially, $\deg_{\Gamma_X(\tau)}(G) \geq |G|$.*

Proof. Since G is a vertex of the graph $\Gamma_X(\tau)$, we have $G \neq \emptyset$. Let $\{p_{i_1}, p_{i_2}, ..., p_{i_s}\}$ be a non-empty subset of G. Since the topology is T_1, the finite set $\{p_{i_1}, p_{i_2}, ..., p_{i_s}\}$ is a closed set and hence, $\{p_{i_1}, p_{i_2}, ..., p_{i_s}\}^c \in \tau$. Note that $\{p_{i_1}, p_{i_2}, ..., p_{i_s}\}^c \notin \{\emptyset, X\}$ and $\{p_{i_1}, p_{i_2}, ..., p_{i_s}\}^c \cup G = X$. This means that the vertex G is adjacent to the vertex $\{p_{i_1}, p_{i_2}, ..., p_{i_s}\}^c$. Since the set G has $2^{|G|}$ subsets and one of these subsets is the emptyset (which is not a vertex of $\Gamma_X(\tau)$), the results directly follows. □

Proposition 2.5. *Let X be an arbitrary set with at least two elemnts and endowed with discrete topology. Then, for each vertex G in $\Gamma_X(\tau)$ we have $\deg_{\Gamma_X(\tau)}(G) = 2^{|G|} - 1$.*

Proof. Note that $\tau = P(X)$ which is the power set of X. Let $H \in \tau$ be a vertex of $\Gamma_X(\tau)$ which is adjacent to G. Thus, $G \cup H = X$ and hence, $G^c \subseteq H$. Let $\hat{H} = H \setminus G^c$ which implies that $\hat{H} \subseteq G$. Since $H \neq X$, we have $\hat{H} \neq G$. Thus, $\hat{H} \in P(G) \setminus \{G\}$. This means that each neighbour H of G in $\Gamma_X(\tau)$ corresponds to a member $\hat{H}$ of $P(G) \setminus \{G\}$ and vice verca. Therefore,

$$\deg_{\Gamma_X(\tau)}(G) = |P(G) \setminus \{G\}| = 2^{|G|} - 1,$$

which completes the proof. □

Corollary 2.6. *Let (X, τ) be a T_1 topological space. Then, the maximum degree and the mnimum degree of the graph $\Gamma_X(\tau)$ is determined as follows.*

i) when X is a finite set. Then, $\Delta\big(\Gamma_X(\tau)\big) = 2^{|X|-1} - 1$ and $\delta\big(\Gamma_X(\tau)\big) = 1$.

ii) when X is an infinite set. Then, $\Delta\big(\Gamma_X(\tau)\big) = \infty$ and $\delta\big(\Gamma_X(\tau)\big) = 2^k - 1$ in which $k = \inf\{\,|G| : G \in \tau,\ G \neq \emptyset\}$.

Proof. At first, assume that X is a finite set and let $x \in X$. Note that since X is a finite set and the topology is T_1, it is a discrete topology, i.e. $\tau = P(X)$, the power set of X. Thus, by considering Proposition 2.3 and Proposition 2.5, we see that

$$\delta(\Gamma_X(\tau)) = \deg_{\Gamma_X(\tau)}(\{x\}) = 2^1 - 1 = 1,$$

and

$$\Delta(\Gamma_X(\tau)) = \deg_{\Gamma_X(\tau)}(X \setminus \{x\}) = 2^{|X|-1} - 1.$$

Now assume that X is an infinite set. Let $p \in X$ be an arbitrary point. Then, $\{p\}^c = X \setminus \{p\}$ is a non-trivial open set and obviously it is an infinite set. Now by Propositions 2.3 and 2.4, we see that

$$\Delta(\Gamma_X(\tau)) = \deg_{\Gamma_X(\tau)}(X \setminus \{p\}) \geq |X \setminus \{p\}| = \infty.$$

For the minimum degree, if each vertex G of $\Gamma_X(\tau)$ is an infinite (open) set, then Proposition 2.4 implies that $\deg_{\Gamma_X(\tau)}(G) = \infty$ and hence,

$$\delta\big(\Gamma_X(\tau)\big) = \infty = \inf\{\,|G| : G \in \tau,\ G \neq \emptyset\}.$$

Otherwise, there exists at least one finite and non-trivial open set in the topology (a vertex in $\Gamma_X(\tau)$) and hence, we can define the integer k as

$$k = \min\{\,|G| : G \in \tau,\ G \neq \emptyset\} = \inf\{\,|G| : G \in \tau,\ G \neq \emptyset\}.$$

Now Proposition 2.3 implies that $\delta\big(\Gamma_X(\tau)\big) = \deg_{\Gamma_X(\tau)}(G)$ in which G is a member of τ with $|G| = k$. □

Now, we state the following corollaries.

Corollary 2.7. *If X is an arbitrary non-empty finite set with the discrete topology τ, then the size of the graph $\Gamma_X(\tau)$ is given by*

$$|E(\Gamma_X(\tau))| = \frac{1}{2}\left(3^{|X|} - 2^{|X|+1} + 1\right).$$

Proof. By using Proposition 2.5, the binomial expansion $(a+b)^n = \sum_{i=0}^{n}\binom{n}{i}a^i b^{n-i}$ and the Handshakin Lemma (the sum of the vertex degrees equals twice the number of edges in each graph), we see that

$$\begin{aligned}
|E(\Gamma_X(\tau))| &= \frac{1}{2}\sum_{G\in P(X)\setminus\{\emptyset,X\}} \deg_{\Gamma_X(\tau)}(G)\\
&= \frac{1}{2}\sum_{G\in P(X)\setminus\{\emptyset,X\}} (2^{|G|}-1)\\
&= \frac{1}{2}\sum_{k=1}^{|X|-1}\binom{|X|}{k}(2^k-1)\\
&= \frac{1}{2}\left(\sum_{k=0}^{|X|}\binom{|X|}{k}(2^k-1) - \binom{|X|}{0}(2^0-1) - \binom{|X|}{|X|}(2^{|X|}-1)\right)\\
&= \frac{1}{2}\left(\sum_{k=0}^{|X|}\binom{|X|}{k}2^k - \sum_{k=0}^{|X|}\binom{|X|}{k}1 - 0 - (2^{|X|}-1)\right)\\
&= \frac{1}{2}\left((2+1)^{|X|} - (1+1)^{|X|} - (2^{|X|}-1)\right)\\
&= \frac{1}{2}\left(3^{|X|} - 2^{|X|+1} + 1\right).
\end{aligned}$$

The proof is complete. □

Corollary 2.8. *Let X be an arbitrary non-empty finite set with the discrete topology. Then the first Zagreb index of $\Gamma_X(\tau)$ is equal to*

$$5^{|X|} - 4^{|X|} - 2(3^{|X|}) + 3(2^{|X|}) - 1.$$

Proof. Again, by Proposition 2.5 and by the definition of the first Zagreb index, we obtain

$$\begin{aligned}
Z_1(\Gamma_X(\tau)) &= \sum_{G\in P(X)\setminus\{\emptyset,X\}} \left(\deg_{\Gamma_X(\tau)}(G)\right)^2 \\
&= \sum_{G\in P(X)\setminus\{\emptyset,X\}} \left(2^{|G|}-1\right)^2 \\
&= \sum_{k=1}^{|X|-1} \binom{|X|}{k} \left(2^k-1\right)^2 \\
&= \sum_{k=0}^{|X|} \binom{|X|}{k} \left(2^k-1\right)^2 - \binom{|X|}{0}\left(2^0-1\right)^2 - \binom{|X|}{|X|}\left(2^{|X|}-1\right)^2 \\
&= \sum_{k=0}^{|X|} \binom{|X|}{k} \left(2^{2k}-2\times 2^k+1\right) - \left(2^{|X|}-1\right)^2 \\
&= (4+1)^{|X|} - 2\times(2+1)^{|X|} + (1+1)^{|X|} - \left(2^{|X|}-1\right)^2 \\
&= 5^{|X|} - 2\times 3^{|X|} - 4^{|X|} + 3\times 2^{|X|} - 1.
\end{aligned}$$

□

Proposition 2.9. *Assume that (X,τ) is a T_1 topological space. If X is infinite, then the Wiener index of $\Gamma_X(\tau)$ is also infinite. Otherwise, the Wiener index of $\Gamma_X(\tau)$ is equal to*

$$2\binom{2^{|X|}-2}{2} - 2^{|X|-1} + 1.$$

Proof. When X is infinite, then for each pair of distinct points $p,q\in X$, two vertices $\{p\}^c$ and $\{q\}^c$ are adjacent in $\Gamma_X(\tau)$. Hence, $|E(\Gamma_X(\tau))|$ is infinite and then the fact $W(\Gamma_X(\tau)) \geq |E(\Gamma_X(\tau))|$ implies that the Wiener index of $\Gamma_X(\tau)$ is infinite. Now suppose that X is a finite set and hence, $\tau = P(X)$ is the discrete topology. Assume that $X = \{x_1, x_2, ..., x_n\}$. Note that $|P(X)| = 2^n$ and hence, the graph $\Gamma_X(\tau)$ has $2^n - 2$ vertices. For the computation of the the Wiener index of $\Gamma_X(\tau)$ we must consider the distance of $\binom{2^n-2}{2}$ (non-ordered) pairs of vertices. By the definition of $\Gamma_X(\tau)$, we have $d_{\Gamma_X(\tau)}(G,H) = 1$ if and only if $G\cup H = X$. By Corollary 2.7, there are

$$\frac{1}{2}\left(3^n - 2^{n+1} + 1\right)$$

(non-ordered) pairs of vertices in $\Gamma_X(\tau)$ whose distance is equal to 1.
Now let G and H be two non-adjacent vertices in $\Gamma_X(\tau)$, and hence, $G\cup H \neq X$.

If $G\cap H \neq \emptyset$, then there exists a point $x\in G\cap H$. Then, we have $G\cup\{x\}^C = X = H\cup H$ which shows that $\{x\}^c$ is a common neighbor of G and H. This implies that $d_{\Gamma_X(\tau)}(G,H) = 2$. The converse is true too. Indeed, from the equation $d_{\Gamma_X(\tau)}(G,H) = 2$ we deduce that G and H are non-adjacent (hence, $G\cup H \neq X$) and there exists some common neighbor A for them. Thus, $G\cup A = X = H\cup A$ implies that $A^c \subseteq G$ and $A^c \subseteq H$. Thus, $A^c \subseteq G\cap H$ which implis that $G\cap H \neq \emptyset$.

If $G \cap H \neq \emptyset$, then for two arbitrary (distinct) points $x \in G$ and $y \in H$ we have

$$G \cup \{x\}^c = \{x\}^c \cup \{y\}^c = \{y\}^c \cup H = X,$$

which implies that $d_{\Gamma_X(\tau)}(G,H) = 3$. Therefore, for each pair of vertices G, H in $\Gamma_X(\tau)$ we have $d_{\Gamma_X(\tau)}(G,H) \leq 3$. Specially, we have $d_{\Gamma_X(\tau)}(G,H) = 3$ if and only if $G \cup H \neq X$ and $G \cap H = \emptyset$. Let G, H be two vertices in $\Gamma_X(\tau)$ whose distance is 3 and hence, $G \cup H \neq X$ and $G \cap H = \emptyset$. Let $K = G \cup H$ and $k = |K|$. Since $G \neq \emptyset$ and $H \neq \emptyset$, we deduce $k = |K| \geq 2$. Also, $G \cup H \neq X$ implies that $k \leq n-1$. Hence, $2 \leq k \leq n-1$. Note that $K = G \cup H$ and $G \cap H = \emptyset$ indicates that $G \cup H$ is a partition of K into two parts. By considering the Stirling numbers of the second type, it is well known that $S(k,2) = 2^{k-1} - 1$. Also, note that there are $\binom{n}{k}$ subsets of X with cardinality k. Thus, the number of the pairs of vertices with distance 3 in $\Gamma_X(\tau)$ is equal to

$$\begin{aligned}
\sum_{k=2}^{n-1} \binom{n}{k} S(k,2) &= \sum_{k=2}^{n-1} \binom{n}{k} (2^{k-1} - 1) \\
&= \sum_{k=0}^{n} \binom{n}{k} (2^{k-1} - 1) \ - \binom{n}{0} (2^{-1} - 1) - \binom{n}{1} (2^0 - 1) - \binom{n}{n} (2^{n-1} - 1) \\
&= \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} 2^k - \sum_{k=0}^{n} \binom{n}{k} + \frac{1}{2} - (2^{n-1} - 1) \\
&= \frac{1}{2} \big(3^n - 3(2^n) + 3\big).
\end{aligned}$$

Hence, the number of the pairs of vertices with distance 2 in $\Gamma_X(\tau)$ is equal to

$$\binom{2^n - 2}{2} \ - \frac{1}{2}\Big(3^n - 2^{n+1} + 1\Big) \ - \frac{1}{2}(3^n - 3(2^n) + 3) = \binom{2^n - 2}{2} - \Big(3^n - 5(2^{n-1}) + 2\Big).$$

Therefore, for the Wiener index of $\Gamma_X(\tau)$ we have

$$\begin{aligned}
W(\Gamma_X(\tau)) &= \frac{1}{2}\Big(3^n - 2^{n+1} + 1\Big) \times 1 + \left(\binom{2^n - 2}{2} - \Big(3^n - 5(2^{n-1}) + 2\Big)\right) \times 2 \\
&\quad + \frac{1}{2}(3^n - 3(2^n) + 3) \times 3 \\
&= 2\binom{2^n - 2}{2} - 2^{n-1} + 1.
\end{aligned}$$

The proof is complete. □

Theorem 2.10. *Let (X, τ) be a T_1 topological space. Then, the chromatic number of the graph is given by $\chi\big(\Gamma_X(\tau)\big) = |X|$.*

Proof. For each vertex G in $\Gamma_X(\tau)$, we have $G \neq X$ and hence, there exists a point $p_G \in X$ such that $p_G \notin G$. By the Axiom of choice (applied for the sets G^c), for each $G \in \tau \setminus \{\emptyset, X\}$ we can select a point p_G with the condition $p_G \notin G$. Since X is T_1, for each $p \in X$ we have

$\{p\}^c \in \tau \setminus \{\emptyset, X\}$ and hence, for $G = \{p\}^c$ the only possible choice for $p_{_G}$ is p. Thus, each point of X will be selected for at least one vertex of $\Gamma_{_X}(\tau)$. Now define the (coloring) function $f : V\big(\Gamma_{_X}(\tau)\big) \to X$ as $f(G) = p_{_G}$ for each $G \in V\big(\Gamma_{_X}(\tau)\big)$. We show that f is a proper coloring. Let G and H be two adjacent vertices in $\Gamma_{_X}(\tau)$. Thus, we have $G \cup H = X$. Hence, from the fact $p_{_G} \notin G$ we deduce that $p_{_G} \in H$, and from the fact $p_{_H} \notin H$ we deduce that $p_{_H} \in G$. Since, $p_{_G} \notin G$ and $p_{_H} \notin H$, we have $p_{_G} \neq p_{_H}$ and this means that $f(G) \neq f(H)$. Therefoe, f is a proper coloring and hence, $\chi\big(\Gamma_{_X}(\tau)\big) \leq |X|$. The set $\{\ \{p\}^c :\ p \in X\}$ induces a clique in $\Gamma_{_X}(\tau)$ because for each pair of distinct points p and q we have $\{p\}^c \cup \{q\}^c = X$. Thus, in each proper coloring of $\Gamma_{_X}(\tau)$ all of the colors assigned to these vertices must be distinct and hence, $\chi\big(\Gamma_{_X}(\tau)\big) \geq |X|$. Thus, we have $\chi\big(\Gamma_{_X}(\tau)\big) = |X|$ and the proof is complete. □

Theorem 2.11. *Assume that (X, τ) is a T_1 topology. Then, $\Gamma_{_X}(\tau)$ is a planar graph if and only if $2 \leq |X| \leq 3$.*

Proof. If $|X| \geq 5$, then the set $\{\ \{p\}^c :\ p \in X\}$ induces a complete graph with at least 5 vertices and hence, K_5 is a subgraph of $\Gamma_{_X}(\tau)$ which implies that $\Gamma_{_X}(\tau)$ is not planar. Hence, assume that $|X| \leq 4$. Thus, (X, τ) is a finite T_1 topology and hence, it is a discrete topology. For two cases $|X| = 2$ and $|X| = 3$, let $X = \{a, b\}$ and $X = \{a, b, c\}$, respecively. Then, the corresponding graph of these two cases are deficted in Figure 1 and obviously these graphs are planar.

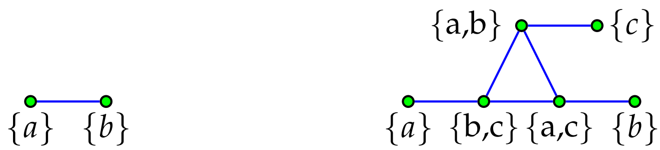

Figure 1.

For the remaining case $|X| = 4$, let $X = \{a, b, c, d\}$ and note that $\tau = P(X)$. Now Consider the graph $\Gamma_{_X}(\tau)$ as shown in Figure 2. Let

$$u_1 = \{a, c\},\ u_2 = \{a, c, d\},\ u_3 = \{a, d\},\ \ v_1 = \{b, d\},\ v_2 = \{b, c, d\},\ v_3 = \{a, b, d\},$$

and consicer the folloing 9 paths in $\Gamma_{_X}(\tau)$:

$$\begin{gathered} u_1 \sim v_1 \ \ ,\ \ u_1 \sim v_2 \ \ ,\ \ u_1 \sim v_3 \\ u_2 \sim v_1 \ \ ,\ \ u_2 \sim v_2 \ \ ,\ \ u_2 \sim v_3 \\ u_3 \sim \{a, b, c\} \sim v_1 \ \ ,\ \ u_3 \sim v_2 \ \ ,\ \ u_3 \sim \{b, c\} \sim v_3 \end{gathered}$$

Two prtite sets of vertices $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ toghether with these 9 paths, provide a subdivision of the complete bipartite graph $K_{3,3}$ in $\Gamma_{_X}(\tau)$. This means that $\Gamma_{_X}(\tau)$ is not planar and the proof is complete. □

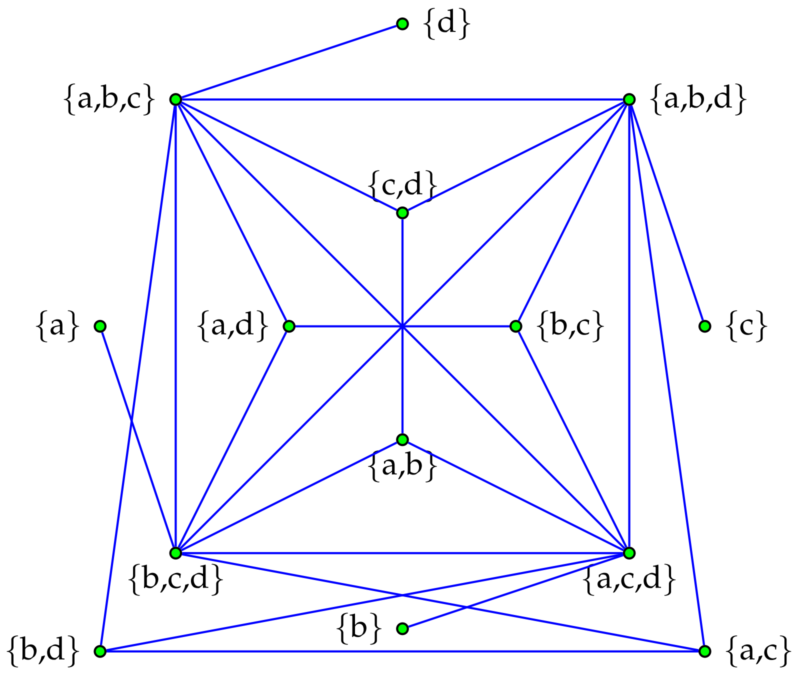

Figure 2. The graph $\Gamma_{_{\{a,b,c,d\}}}(P(\{a,b,c,d\}))$.

Theorem 2.12. *Let (X,τ) be a discrete topological space. Then, for the matching number (the edge independence number) of $\Gamma_X(\tau)$ we have $\alpha'(\Gamma_X(\tau)) = 2^{|X|-1} - 1$.*

Proof. Since $\tau = P(X)$, for each $A \subseteq X$ we have $A \in \tau$ and $A^c \in \tau$. Note that $A \cup A^c = X$ for each $A \notin \{\emptyset, X\}$ and hence, two vertices A and A^c are adjacent in $\Gamma_X(\tau)$. The cardinality of $P(X)$ is $2^{|X|}$ and all of it's members can be partioned into $\frac{2^{|X|}}{2}$ complement pairs like A, A^c in which one of these pairs is $\emptyset, X$. Thus, the vertex set of $\Gamma_X(\tau)$ can be partioned into $\frac{2^{|X|}}{2} - 1$ adjacent pairs of vertices which are. This determines a perfect matching and the result follows. □

Theorem 2.13. *Let (X,τ) be an arbitrary topological space. Then, for the chromatic index (the edge chromatic number) of $\Gamma_X(\tau)$ we have $\chi'(\Gamma_X(\tau)) \leq 2^{|X|-1} - 1$. Moreover, when (X,τ) is a discrete topological space, then $\chi'(\Gamma_X(\tau)) = 2^{|X|-1} - 1$ and hence, the mentioned upper bound is sharp.*

Proof. Consider the power set $P(X)$ of X and let

$$M = \big\{\{A, A^c\} : \ A \in P(X),\ A \neq \emptyset,\ A \neq X\big\}.$$

Note that $|M| = 2^{|X|-1} - 1$. Now define (the edge coloring function) $f : E\big(\Gamma_X(\tau)\big) \to M$ as $f(GH) = \{G \cap H, (G \cap H)^c\}$ for each edge GH in $\Gamma_X(\tau)$. We show that f is a (proper) edge coloring. Let GH and GK be two distinct and adjacent edges in $\Gamma_X(\tau)$. Thus, we have $H \neq K$, $G \cup H = X$ and $G \cup K = X$. Since $G \cup H = X$, we have $G^c \subseteq H$ and similarly $G^c \subseteq H$. Hence, $H = G^c \cup H_1$ and $K = G^c \cup K_1$ in which $H_1 \subseteq G$ and $K_1 \subseteq G$. Note that,

$$G \cap H = H_1,\ G \cap K = K_1,$$

and

$$f(GH) = \{G \cap H, (G \cap H)^c\} = \{H_1, H_1^c\}, \;\; f(GK) = \{G \cap K, (K \cap K)^c\} = \{K_1, K_1^c\}.$$

Suppose (on the contrary) that we have $f(GH) = f(GK)$. Then, we must have

$$\{H_1, H_1^c\} = \{K_1, K_1^c\}.$$

Note that $H_1 \in \{H_1, H_1^c\}$ and hence $H_1 \in \{K_1, K_1^c\}$. Since $K_1 \subseteq G$, we have $K_1^c \supseteq G^c \neq \emptyset$. Thus, the fact $\emptyset \neq H_1 \subseteq G$ implies that $H_1 \neq K_1^c$ and hence, $H_1 = K_1$. Therefore, we have

$$G \cup H = X = G \cup K \; , \; G \cap H = H_1 = K_1 = G \cap K.$$

Let $h \in H$. if $h \in G$, then $h \in H \cap G = K \cap G$ and hence, $h \in K$. Otherwise, $h \notin G$ and the fact $h \in H \subseteq H \cup G = K \cup G$ implies that $h \in K$. In each case, we have $h \in K$ and hence, $H \subseteq K$. Similarly, we obtain $K \subseteq H$. Thus, $H = K$, which is a contradiction. This contradiction shows that $f(GH) = f(GK)$ is impossible and hence, f is a proper edge coloring of $\Gamma_X(\tau)$ with the color set M. Thus, $\chi'(\Gamma_X(\tau)) \leq |M| = 2^{|X|-1} - 1$.

Now let τ be the discrete topology, i.e. $\tau = P(X)$. Note that in each proper (and optimal) edge coloring, adjacent edges must ereceived different colors and hence, $\chi'\big(\Gamma_X(\tau)\big) \geq \Delta\big(\Gamma_X(\tau)\big)$. By Theorem 2.6, the maximum degree of $\Gamma_X(\tau)$ is $2^{|X|-1} - 1$. Therefore, we have $\chi'\big(\Gamma_X(\tau)\big) \geq 2^{|X|-1} - 1$ and hence, $\chi'\big(\Gamma_X(\tau)\big) = 2^{|X|-1} - 1$ in this case. □

Theorem 2.14. *Let (X, τ) be a topological space. Then, the following statements hold.*

- *i) $\Gamma_X(\tau)$ is a path if and only if $\tau = \{\emptyset, A, X\}$ or $\tau = \{\emptyset, A, A^c, X\}$ for some $\emptyset \neq A \subsetneq X$.*
- *ii) $\Gamma_X(\tau)$ is a complete graph if and only if $\tau = \{\emptyset, A, X\}$ or $\tau = \{\emptyset, A, A^c, X\}$ in which $A \notin \{\emptyset, X\}$.*
- *iii) $\Gamma_X(\tau)$ is a complete bipartite graph if and only if $\tau = \{\emptyset, A, A^c, X\}$ for some $A \notin \{\emptyset, X\}$, i.e. $\Gamma_X(\tau) = K_{1,1}$.*
- *iv) If τ is a linearly ordered set (with respect to the inclusion relation $\subseteq$), then $\Gamma_X(\tau)$ is an empty graph (i.e., a graph with no edge).*

Proof. At first, we prove (i). It is easy to see that for $\tau = \{\emptyset, A, X\}$ the graph $\Gamma_X(\tau)$ is the one vertex path P_1, and for $\tau = \{\emptyset, A, A^c, X\}$ the graph $\Gamma_X(\tau)$ is the two vertex path P_2. Now let GH be an arbitrary edge in $\Gamma_X(\tau)$. Note that $G \cap H \subsetneq G$ and $G \cap H \subsetneq H$. If $G \cap H \neq \emptyset$, then $G \cap H$ is a vertex of $\Gamma_X(\tau)$. Hence, when $G \cap H$ is not adjacent to any vertex, which means that $\Gamma_X(\tau)$ contains an isolated vertex and it is not a path. Also, when $G \cap H$ is adjacent to a vertex K, then $K \cup (G \cap H) = X$ and hence, $K \cup G = X = K \cup H$ which means K is adjacent to both of vertices G and H. Thus, $\Gamma_X(\tau)$ has a triangle on three vertces G, H, K and hence, $\Gamma_X(\tau)$ is not a path. These facts imply that when $\Gamma_X(\tau)$ is a path, for each pair of adjacent vertices G and H we must have $G \cap H = \emptyset$. Now assume that $\Gamma_X(\tau)$ is a path. If the number of vertices of $\Gamma_X(\tau)$ is 1 or two, then obviously we have $\tau = \{\emptyset, A, X\}$ or $\tau = \{\emptyset, A, A^c, X\}$

for some $\emptyset \neq A \subsetneq X$. Hence, assume (on the contrary) that $\Gamma_X(\tau)$ is a path with at least 3 vertices. Let G, H, K be three (consequtive) vertices in $\Gamma_X(\tau)$, i.e. GH and HK are two edges of $\Gamma_X(\tau)$. Thus, $G \cup H = X$ and $H \cup K = X$. By the previous result, we must have $G \cap H = \emptyset$ and $H \cap K = \emptyset$. Now $G \cup H = X$ and $G \cap H = \emptyset$ imply that $G = H^c$, and similarly, $H \cup K = X$ and $H \cap K = \emptyset$ imply that $K = H^c$. Thus, $G = H^c = K^c$, which is a contradiction. The proof of (i) is complete.

Now we prove (ii) and (iii). At first, note that $K_2 = K_{1,1}$.
It is easy to check that when $\tau = \{\emptyset, A, X\}$ or $\tau = \{\emptyset, A, A^c, X\}$ for some $A \notin \{\emptyset, X\}$, then the graph $\Gamma_X(\tau)$ is the complete graph K_1 or K_2, respectively. Assume that G and H are two adjacent vertices in $\Gamma_X(\tau)$. If $G \cap H \neq \emptyset$, then the vertex $G \cap H$ is non-adjacent to both of vertices G and H which implies that $\Gamma_X(\tau)$ neither is a complete graph nor a complete bipartite graph. If $G \cap H = \emptyset$, then we have $H = G^c$. Therefore, when $\Gamma_X(\tau)$ is a complete graph or a complete bipartite graph, for each pair of adjacent vertices G and H we have $H = G^c$. This implies that the connected graph $\Gamma_X(\tau)$ has two vertices, i.e. $\Gamma_X(\tau) = K_2$.
Finally, note that when $A \subsetneq B \neq X$, then $A \cup B = B \neq X$. Hence, each pair of vertices in $\Gamma_X(\tau)$ are non-adjacent and hence, $\Gamma_X(\tau)$ is an empty graph. This proves (iv). □

3 Conclusions

In this research we define a simple graph on a general topological space whose vertices are non-trivial open sets and two vertices are adjacent just when their union become the whole points set. Some graphical parameters like clique and chromatic numbers are determined and its planarity is completely determined. Further works can be focused on special topological spaces or other graphical parameters.

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Conflicts of Interests

The authors declare that they have no competing interests.

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