Journal of Discrete Mathematics and Its Applications 10 (2) (2025) 161–182

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Research Paper

Generalized k-plane trees

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Academic Editor: Hamidreza Maimani

Abstract. Plane trees and noncrossing trees have been generalized by assigning labels to the vertices from a given set such that a prior coherence condition is satisfied. These trees are called k-plane trees and k-noncrossing trees respectively if k labels are used. Results of plane trees and noncrossing trees were recently unified by considering d-dimensional plane trees where plane trees are 1-dimensional plane trees and noncrossing trees are 2-dimensional plane trees. In this paper, d-dimensional k-plane trees are introduced and enumerated according to number of vertices and label of the root, root degree, number of components constituting a forest, label of the eldest child of the root and the length of the leftmost path. The equivalent results for plane trees and noncrossing trees follow easily from our results as corollaries.

Keywords. k-plane tree, k-noncrossing tree, d-dimensional k-plane tree, root degree, forest, eldest child, leftmost path.

Mathematics Subject Classification (2020): 05C15, 05C10.

1 Introduction and preliminary result

Combinatorial structures counted by Catalan and Catalan-like numbers have been studied for more than a century. Among these structures counted by Catalan numbers are plane trees as given in sequence A000108 of the celebrated online encyclopaedia of integer sequence [30]. Stanley provided a list and interconnections among these structures in [32].

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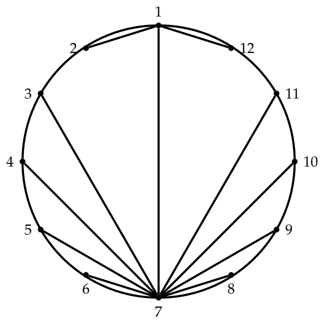
Received 16 March 2025; Revised 25 March 2025; Accepted 24 April 2025

First Publish Date: 01 June 2025

Formally, a *plane tree* is a tree (rooted) drawn in the plane such that all the vertices are ordered [1]. Noncrossing trees are among the discrete structures counted by Catalan-like numbers (also called Fuss-Catalan numbers). These are trees drawn in the plane such that vertices are on the circumference of a circle and edges do not cross inside the circle [12]. Since noncrossing trees are drawn in the plane then they are also plane trees. In 2002, Panholzer and Prodinger [29] introduced (l,r)-representation of noncrossing trees, which is a way of representing noncrossing trees as plane trees. If (i,j) is an edge in a noncrossing tree such that $i<j$ (respectively, $i>j$) and i is closer to the root than j then vertex j is an *ascent* (respectively, a *descent*). Let (i,j) be an edge in a path from the root of a noncrossing tree such that j is an ascent (respectively, a descent) then j is represented by letter r (respectively, l) in the plane tree representing the noncrossing tree where vertex 1 is the root of the plane tree. The plane tree representing noncrossing tree in the (l,r)-representation has all non-root vertices labeled by l or r. Of course, all vertices adjacent to the root are labeled by r. In Figure 1, we provide a noncrossing tree together with its (l,r)-representation.



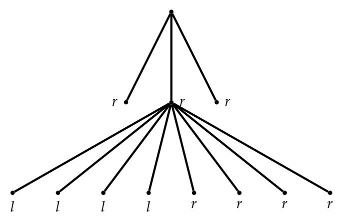


Figure 1. A noncrossing tree on 12 vertices and its (l,r)-representation.

Let (i,j) be an edge in a plane tree or (l,r)-representation of a noncrossing tree with the condition that i is closer to the root than j. The vertex j is thus the *child* of i. This means that i is the *parent* of j. A vertex is said to reside on *level* ℓ if there are ℓ edges on the path from the root to that particular vertex. It is worth remarking that the root resides on level 0 since there is no edge from the root to itself. The vertex that appears on the far left of all other vertices that reside on a given level is the *eldest child* on that level. A path from the root to a given eldest child such that all other vertices on the path are eldest children is a *leftmost path*. The *degree* of a vertex in a plane tree is the number of children of that vertex. A vertex of degree 0 which is not a root is a *leaf* and a non-leaf vertex is an *internal vertex*. Leaves together with roots of degree 1 are *endpoints*. A plane tree in which each internal vertex has at most t children is a t-ary tree. Setting $t=2$, we find binary trees counted by Catalan numbers and for $t=3$, we get ternary trees which are enumerated by the same formula for the number of noncrossing trees. A sequential arrangement of degrees of all vertices in a plane tree is called *degree sequences.* A *forest* is a graph with a finite number of components such that each component is a tree. Plane trees have been counted using statistics such as number of vertices [1], root degree, leaves [2], level at which a vertex of a given degree resides [2, 5], degreee sequence [4] and forests [31] among other parameters.

In the past three-quarter century, plane trees have been generalized by considering their block graphs [9,22,27] or by considering plane trees in which the vertices receive labels from a given set such that the sum of labels of endpoints of each edge satisfies a certain coherence condition [7, 8, 10, 14, 23, 26]. The latter case is the most relevant in the present study. In [7], Gu and Prodinger generalized plane trees to 2-*plane trees*. These are plane trees in which the nodes are labeled 1 or 2 such that the sum of labels of adjacent vertices does not exceed 3. They found a counting formula for these trees given the number of vertices and label of the root. The work was extended in [8] by Gu, Prodinger and Wagner to k-*plane trees* which are plane trees in which the vertices are labeled with integers in the set $\{1, 2, \ldots, k\}$ such that for each edge the sum of labels of its endpoints is no greater than $k + 1$. The aforementioned authors counted the said trees by number of vertices and label of the root. Moreover, they established a bijection between the set of k-plane trees and the set of $(k + 1)$-ary trees. Formulas obtained by Gu, Prodinger and Wagner were further refined by Nyariaro and Okoth [14] to enumerate k-plane trees by root degree and number of forests. Okoth and Wagner obtained counting formulas for k-plane trees by occurrences of vertices of a certain type [26]. Furthermore, Okoth [19] and Nyariaro and Okoth [13] constructed various bijections relating k-plane trees to other combinatorial structures. It is worth noting that 1-plane trees are plane trees. Recently, Oduol, Okoth and Nyamwala [16] introduced the set of k_1-*plane trees* which are k-plane trees in which all vertices labeled 1 must on the left of all others. The authors then enumerated them by number of vertices, root degree, label of the eldest child of the root, length of the leftmost path and number of forests among other statistics.

Noncrossing trees, since their introduction in 1998 by Noy [12], have been enumerated by number of vertices [12], leaves, degree sequence, forests [6], endpoints, maximum degree [3], levels [15] to name but a few. Noncrossing trees have also been generalized considering their block graphs [20, 22] and by labeling the vertices [24, 28, 34]. The latter is key in this work. In 2010, Pang and Lv [28] introduced and enumerated k-noncrossing trees. A k-*noncrossing tree* is a noncrossing tree in which vertices receive labels from the set $\{1, \ldots, k\}$ such that if (i, j) is an ascent on the path from the root then $i + j \leq k + 1$. These authors found the counting formulas for these trees by number of vertices and label of the root. The research was extended by Okoth who enumerated k-noncrossing trees by root degree and number of forests [21]. In addition, Okoth and Wagner [26] enumerated the set of k-noncrossing trees by occurrences of vertices of a certain type. Bijections of k-noncrossing trees have been established in [13]. It is important to note that 1-noncrossing trees are the ordinary noncrossing trees. The set of 2-noncrossing trees, as combinatorial structures, was introduced in 2009 by Yan and Liu [34] and enumerated them by number of vertices and label of the root. The authors also established a relationship between the set of 2-noncrossing trees with root labeled 2 and the set of 5-ary trees. In 2024, Oduol, Okoth and Nyamwala [17] introduced the set of k_1-*noncrossing trees*. A k_1-*noncrossing tree* is a k-noncrossing tree that when represented as a plane tree using the (l, r)-representation has all its ascents and descents labeled 1 coming on the left of all other ascents and descents respectively. The aforesaid authors then enumerated the set of k_1-noncrossing trees by number of vertices, root degree, label of the eldest child of the root, length of the leftmost path and number of forests [17].

One of the celebrated concepts in enumeration of plane trees, noncrossing trees and their block graphs is the butterfly decomposition introduced by Flajolet and Noy [6] in the enumeration of noncrossing trees. A *butterfly* is defined as an ordered pair of noncrossing trees that share a root. So considering any vertex in a noncrossing tree, there is a left wing and a right wing of a butterfly. A plane tree therefore is a noncrossing tree in which for each internal vertex, there is no left wing of the butterfly rooted at that vertex. Okoth and Kasyoki [25] in an attempt to unify results for plane trees and noncrossing trees, considered noncrossing trees in which butterflies rooted at each internal vertex have d wings instead of two wings. They stated that there is only one right wing and the remaining $d-1$ wings are left wings. They coined the name *d-dimensional plane tree* for such a tree. We remark that:

(i) The wings of the d-dimensional plane tree are given labels with the left wing being the first wing and the rightmost wing being the last wing.

(ii) A wing of a butterfly rooted at the root is a right wing.

(iii) A plane treee is a 1-dimensional plane tree and it has no left wing. Moreover, a noncrossing tree is a 2-dimensional plane tree.

We also note that in Figure 2, we get a 3-dimensional plane tree with 12 vertices.

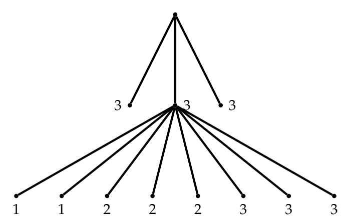


Figure 2. A 3-dimensional plane tree on 12 vertices where the labels are the labels of the wings.

We now introduce the main objects of our study.

Definition 1.1. *A* d-dimensional k-plane tree *is a noncrossing tree in which butterflies rooted at all internal nodes (if we consider the* (l,r)*-representation of the noncrossing tree) have* d *wings such that if* (i,j) *is an ascent in the right wing then* $i+j \leq k+1$ *and all children of the root are in the right wing.*

Figure 3, is a 3-dimensional 4-plane tree on 12 vertices with root labeled 1.
Oduol, Okoth and Nyamwala [18], have since introduced the set of d-dimensional k_1-plane trees and enumerated them by number of vertices and label of the root, label of the eldest child of the root, root degree, length of the leftmost path and number of forests.

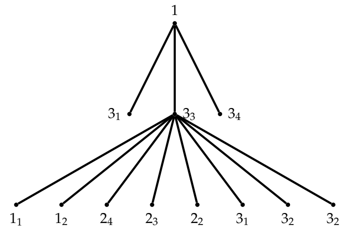


Figure 3. A 3-dimensional 4-plane tree on 12 vertices with root labeled 1 where i_j represents vertex labeled j in wing i.

Definition 1.2 ([18]). *A* d-dimensional k_1-plane tree *is a d-dimensional k-plane tree in which all descents and ascents labeled* 1 *must be on the left of all others.*

Consider the set of d-dimensional k-plane trees. If one of the endpoints of an ascent edge (in the first wing) in d-dimensional k-plane tree is labeled by i then the other endpoint must have a label no more than $k-i+1$. Let $P_i(x) = P_i$ be the generating function for d-dimensional k-plane trees with roots labeled by i where x marks a node.
Then,

$$P_i(x) = x \cdot \frac{1}{1 - \frac{P_1^{d-1}}{x^{d-1}}(P_1 + P_2 + \cdots + P_{k-i+1})}. \tag{1}$$

We now look for suitable substitutions to solve the system of the functional equations (1). Let $P_i(x) = (\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{i-1}$ and $x = (\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{k}$. From (1) we have,

$$\begin{aligned}
P_i(x) &= x \cdot \frac{1}{1 - \frac{((\sqrt[d]{x})^{d-1}\sqrt[d]{y})^{d-1}}{x^{d-1}}(\sqrt[d]{x})^{d-1}\left(\sqrt[d]{y} + \sqrt[d]{y}(1-y) + \cdots + \sqrt[d]{y}(1-y)^{k-i}\right)} \\
&= x \cdot \frac{1}{1 - \frac{x^{d-1}y}{x^{d-1}}\left(1 + (1-y) + \cdots + (1-y)^{k-i}\right)} = x \cdot \frac{1}{1 - \left(1 - (1-y)^{k-i+1}\right)} \\
&= x \cdot \frac{1}{(1-y)^{k-i+1}} = (\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{k} \cdot \frac{1}{(1-y)^{k-i+1}} = (\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{i-1}.
\end{aligned}$$

Since the substitutions $P_i(x) = (\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{i-1}$ and $x = (\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{k}$ satisfy (1) and the second equation does not depend on i then these are the rights substitutions to solve the system of functional equations (1). So, $y = x(1-y)^{-kd}$.

The following theorem is key in the extraction of the coefficient of x^n in P_i.

Theorem 1.3 (Lagrange-Bürmann inversion, [33]). *Let $P(x)$ be a generating function that satisfies the functional equation $P(x) = x\phi(P(x))$, where $\phi(0) \neq 0$. Then, $[x^n]S(P(x)) = [p^{n-1}](S'(p)\phi(p)^n)$ where S is any analytic function.*

Now, we apply Lagrange-Bürmann inversion (Theorem 1.3) to extract the coefficient of

x^n in P_i. We have,

$$\begin{aligned}[x^n]P_i =&[x^n](\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{i-1} = [x^{n-(d-1)/d}]\sqrt[d]{y}(1-y)^{i-1}\\ =&\frac{1}{n-\frac{d-1}{d}}[y^{n-1-(d-1)/d}]\left(\frac{1}{d(\sqrt[d]{y})^{d-1}}(1-y)^{i-1}-(i-1)\sqrt[d]{y}(1-y)^{i-2}\right)\\ &(1-y)^{-kd(n-(d-1)/d)}\\ =&\frac{1}{d(n-1)+1}[y^{n-1}]\,(1-(d(i-1)+1)y)\,(1-y)^{-k(d(n-1)+1)+i-2}.\end{aligned}$$

By binomial theorem, we obtain

$$\begin{aligned}&[x^n]P_i\\ =&\frac{1}{d(n-1)+1}[y^{n-1}]\,(1-(d(i-1)+1)y)\sum_{a\geq 0}\binom{k(d(n-1)+1)-i+a+1}{a}y^a\\ =&\frac{1}{d(n-1)+1}\left[\binom{k(d(n-1)+1)-i+n}{n-1}-(d(i-1)+1)\binom{k(d(n-1)+1)-i+n-1}{n-2}\right]\\ =&\frac{1}{d(n-1)+1}\cdot\frac{(k-i+1)(d(n-1)+1)}{k(d(n-1)+1)-i+n}\binom{k(d(n-1)+1)-i+n}{n-1}\\ =&\frac{k-i+1}{k(d(n-1)+1)-i+1}\binom{(kd+1)(n-1)+k-i}{n-1}.\end{aligned}$$

We summarize the discussions with the following result.

Theorem 1.4. *There are*

$$\frac{k-i+1}{(dk+1)n-k(d-1)-i}\binom{(dk+1)n-k(d-1)-i}{n-1} \tag{2}$$

d-dimensional k-plane trees with n vertices whose root is labeled by i.

2 Consequences of Theorem 1.4

We get the following corollary upon setting $d=1$ in (2).

Corollary 2.1. *The number of k-plane trees with n vertices whose root is labeled by i is given by*

$$\frac{k-i+1}{(k+1)n-i}\binom{(k+1)n-i}{n-1}. \tag{3}$$

Formula (3) was first derived by Gu, Prodinger and Wagner in [8].

Corollary 2.2 ([26]). *The number of k-noncrossing trees with n vertices whose root is labeled by i is given by*

$$\frac{k-i+1}{(2k+1)n-k-i}\binom{(2k+1)n-k-i}{n-1}.$$

Proof. Set $d = 2$ in (2). □

Upon setting $i = 1$ in (2), we find that there are

$$\frac{k}{(dk+1)n - k(d-1) - 1}\binom{(dk+1)n - k(d-1) - 1}{n-1} \tag{4}$$

d-dimensional k-plane trees on n vertices whose root is labeled 1.

By setting $d = 1$ and $d = 2$ in (4), we rediscover the formulas for the number of k-plane trees and k-noncrossing trees on n vertices whose root is labeled 1 which were obtained in [8] and [26] respectively. On the other hand, setting $i = k$ in (2), we get

$$\frac{1}{n-1}\binom{(dk+1)(n-1)}{n-2}, \tag{5}$$

as the formula for the number of d-dimensional k-plane trees on n vertices with root labeled by k. Consequently, setting $d = 1$ in (5), we find

$$\frac{1}{n-1}\binom{(k+1)(n-1)}{n-2} \tag{6}$$

as the number of k-plane trees on n vertices with the root labeled by k. Formula (6) was also derived by Gu, Prodinger and Wagner in [8]. Moreover, setting $d = 2$ in (5), we obtain

$$\frac{1}{n-1}\binom{(2k+1)(n-1)}{n-2} \tag{7}$$

as the number of k-noncrossing trees on n vertices with the root labeled by k, a formula that was also obtained by Okoth and Wagner in [26]. The formula,

$$\frac{1}{d(n-1)+1}\binom{(d+1)(n-1)}{n-1},$$

counts d-dimensional plane trees on n vertices. The formula is arrived at upon letting $k = 1$ in (5) and it was also obtained by Okoth and Kasyoki in [25] where the cases with $d = 1$ and $d = 2$ were obtained earlier by Dershowitz and Zaks in [2] and Noy in [12] respectively.

Corollary 2.3. *The total number of d-dimensional k-plane trees on n vertices is given by*

$$\frac{1}{n-1}\binom{(dk+1)(n-1)}{n} - \frac{d-1}{d(n-1)+1}\binom{(dk+1)n - k(d-1) - 1}{n}. \tag{8}$$

Proof. We extract the coefficient of x^n in $P_1 + P_2 + \cdots + P_k$.

$$\begin{aligned}
&[x^n](P_1 + P_2 + \cdots + P_k) \\
&= [x^n]\left((\sqrt[d]{x})^{d-1}\sqrt[d]{y} + (\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y) + \cdots + (\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{k-1}\right) \\
&= [x^n](\sqrt[d]{x})^{d-1}\sqrt[d]{y}\left(1 + (1-y) + \cdots + (1-y)^k\right) \\
&= [x^{n-1+1/d}]y^{1/d-1}\left(1 - (1-y)^k\right).
\end{aligned}$$

Now, by Lagrange-Burmann inversion, we get

$$
\begin{aligned}
&[x^n](P_1+P_2+\cdots+P_k)\\
&=\frac{1}{n-1+1/d}[y^{n-2+1/d}]\left(ky^{1/d-1}(1-y)^{k-1}-(1-1/d)y^{1/d-2}(1-(1-y)^k)\right)\\
&\quad(1-y)^{-dk(n-1+1/d)}\\
&=\frac{1}{d(n-1)+1}[y^n]\left(dky(1-y)^{k-1}-(d-1)(1-(1-y)^k)\right)(1-y)^{-k(d(n-1)+1)}\\
&=\frac{1}{d(n-1)+1}[y^n]\left(dky(1-y)^{-(dk(n-1)+1)}-(d-1)(1-y)^{-k(d(n-1)+1)}\right.\\
&\quad\left.+(d-1)(1-y)^{-dk(n-1)}\right)\\
&=\frac{1}{d(n-1)+1}\left[dk\binom{(dk+1)(n-1)}{n-1}-(d-1)\binom{(dk+1)(n-1)+k}{n}\right.\\
&\quad\left.+(d-1)\binom{(dk+1)(n-1)}{n}\right]\\
&=\frac{1}{d(n-1)+1}\left[\frac{dk(d(n-1)+1)}{n}\binom{(dk+1)(n-1)}{n-1}-(d-1)\binom{(dk+1)n-k(d-1)-1}{n}\right]\\
&=\frac{dk}{n}\binom{(dk+1)(n-1)}{n-1}-\frac{d-1}{d(n-1)+1}\binom{(dk+1)n-k(d-1)-1}{n}.
\end{aligned}
$$

□

Setting $d=1$ and $d=2$ in (8), we obtain the following results.

Corollary 2.4. *There are*

$$\frac{1}{n-1}\binom{(k+1)(n-1)}{n}$$

k-plane trees on n vertices.

Corollary 2.5. *There are*

$$\frac{1}{n-1}\binom{(2k+1)(n-1)}{n}-\frac{1}{2n-1}\binom{(2k+1)n-k-1}{n}$$

k-noncrossing trees on n vertices.

By setting $k=1$ and performing simple algebraic manipulations, we arrive at the following corollary.

Corollary 2.6. *The total number of d-dimensional plane trees on n vertices is given by*

$$\frac{1}{d(n-1)+1}\binom{(d+1)(n-1)}{n-1}. \tag{9}$$

We remark that the Fuss-Catalan number (9) was recently obtained by Okoth and Kasyoki in [25], where the case $d=1$, Catalan number, counts plane trees and $d=2$ gives the number of noncrossing trees with n vertices.

3 Root degree

In this section, we obtain counting formulas for d-dimensional k-plane trees in which the label of the root, the labels of its children as well as the degree of the root are stated.

Theorem 3.1. *There are*

$$\frac{(dk-i+1)r}{(dk+1)(n-1)-ir}\binom{(dk+1)(n-1)-ir}{n-r-1} \tag{10}$$

d-dimensional k-plane trees on n vertices with root of degree r and labeled by j such that all children of the root are labeled by i.

Proof. Since the root of each child of the root is labeled by i, then to arrive at our result we extract the coefficient of x^n in $x\left(\frac{P_1^{d-1}P_i}{x^{d-1}}\right)^r$ which is done in the sequel.

$$\begin{aligned}
[x^n]x\left(\frac{P_1^{d-1}P_i}{x^{d-1}}\right)^r &= [x^{n-1}]\left(\frac{P_1^{d-1}P_i}{x^{d-1}}\right)^r \\
&= [x^{n-1}]\left(\frac{((\sqrt[d]{x})^{d-1}\sqrt[d]{y})^{d-1}}{x^{d-1}}\right)^r\left((\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{i-1}\right)^r \\
&= [x^{n-1}]y^r(1-y)^{(i-1)r}.
\end{aligned}$$

By Lagrange-Bürmann inversion, we obtain

$$\begin{aligned}
[x^n]x\left(\frac{P_1^{d-1}P_i}{x^{d-1}}\right)^r =& \frac{1}{n-1}[y^{n-2}]\left(ry^{r-1}(1-y)^{(i-1)r}-(i-1)ry^r(1-y)^{(i-1)r-1}\right) \\
&(1-y)^{-dk(n-1)} \\
=& \frac{r}{n-1}[y^{n-r-1}]\,(1-iy)\,(1-y)^{-(dk(n-1)-(i-1)r+1)} \\
=& \frac{r}{n-1}[y^{n-r-1}]\,(1-iy)\sum_{a\geq 0}\binom{dk(n-1)+a-(i-1)r}{a}y^a.
\end{aligned}$$

So,

$$\begin{aligned}
[x^n]x\left(\frac{P_1^{d-1}P_i}{x^{d-1}}\right)^r &= \frac{r}{n-1}\left(\binom{(dk+1)(n-1)-ir}{n-r-1}-i\binom{(dk+1)(n-1)-ir-1}{n-r-2}\right) \\
&= \frac{(dk-i+1)r}{(dk+1)(n-1)-ir}\binom{(dk+1)(n-1)-ir}{n-r-1}.
\end{aligned}$$

□

With $d=1$ in (10), we get the following result.

Corollary 3.2. *There are*

$$\frac{(k-i+1)r}{(k+1)(n-1)-ir}\binom{(k+1)(n-1)-ir}{n-r-1} \tag{11}$$

k-plane trees on n vertices with root of degree r and labeled by j such that all the children of the root are labeled by i.

On further setting $i = 1$ in (11), we obtain

$$\frac{r}{n-1}\binom{(k+1)(n-1)-r-1}{n-r-1}$$

as the formula for the number of k-plane trees on n vertices with root of degree r and labeled by k. This formula was initially discovered by Okoth and Wagner in [26].

Setting $d = 2$ in (10), we get the formula for the number of k-noncrossing trees with a given root degree.

Corollary 3.3. *The number of k-noncrossing trees on n vertices with root of degree r and labeled by j such that all the children of the root are labeled by i is given by*

$$\frac{(2k-i+1)r}{(2k+1)(n-1)-ir}\binom{(2k+1)(n-1)-ir}{n-r-1}. \tag{12}$$

Also, letting $i = 1$ in (12), we arrive at

$$\frac{r}{n-1}\binom{(2k+1)(n-1)-r-1}{n-r-1}$$

which counts k-noncrossing trees on n vertices with root of degree r and labeled by k.

Setting $i = k$ in (10), we find that there are

$$\frac{((d-1)k+1)r}{(dk+1)(n-1)-kr}\binom{(dk+1)(n-1)-kr}{n-r-1} \tag{13}$$

d-dimensional k-plane trees on n vertices with root of degree r and labeled 1 such that all the children of the root are labeled by k. Moreover, if we set $k = 1$ in (13), then we find that there are

$$\frac{r}{n-1}\binom{(d+1)(n-1)-r-1}{n-r-1} \tag{14}$$

d-dimensional plane trees on n vertices with root of degree r. Formula (14) was recently obtained by Okoth and Kasyoki in [25]. We obtain the following result upon setting $i = 1$ in (10).

Corollary 3.4. *There are*

$$\frac{r}{n-1}\binom{(dk+1)(n-1)-r-1}{n-r-1} \tag{15}$$

d-dimensional k-plane trees on n vertices with root of degree r and labeled by k.

We remark that one can arrive at (14) by setting $k=1$ in (15). If $d=1$ and $d=2$ in (15), we obtain the formulas

$$\frac{r}{n-1}\binom{(k+1)(n-1)-r-1}{n-r-1}$$

and

$$\frac{r}{n-1}\binom{(2k+1)(n-1)-r-1}{n-r-1}$$

which counts k-plane trees and k-noncrossing trees on n vertices with root of degree r and labeled by k that were initially obtained by Okoth and Wagner in [26] and Okoth in [21] respectively.

We generalize Theorem 3.1 in the following theorem.

Theorem 3.5. *There are*

$$\frac{dkr-s}{(dk+1)(n-1)-s-r}\binom{(dk+1)(n-1)-s-r}{n-r-1}\binom{r}{r_1,r_2,r_3,\dots,r_{k-i+1}} \tag{16}$$

d-dimensional k-plane trees on n vertices with root labeled by i such the root has r children, r_j of which are labeled by j where $j=1,2,\dots,k-i+1$ and $s:=r_2+2r_3+\cdots+(k-i)r_{k-i+1}$.

Proof. Let $P_i(x)$ be the univariate generating function for d-dimensional k-plane trees rooted at a node labeled by i, with x marking a vertex. Since there are r_i subtrees labeled by $i=1,2,\dots,k$ which are rooted at the children of the root then the generating function for the desired d-dimensional k-plane trees in which the position of the subtrees is not taken into consideration is

$$\begin{aligned}x\left(\frac{P_1(x)^d}{x^{d-1}}\right)^{r_1}\left(\frac{P_1(x)^{d-1}P_2(x)}{x^{d-1}}\right)^{r_2}\cdots\left(\frac{P_1(x)^{d-1}P_{k-i+1}(x)}{x^{d-1}}\right)^{r_{k-i+1}}\\ =x^{(1-d)r+1}P_1^{r(d-1)}P_1^{r_1}P_2^{r_2}\cdots P_{k-i+1}^{r_{k-i+1}}.\end{aligned}$$

We now proceed to extract the coefficient x^n in the generating function.

$$\begin{aligned}[x^n]x^{(1-d)r+1}P_1^{r(d-1)}P_1^{r_1}P_2^{r_2}\cdots P_{k-i+1}^{r_{k-i+1}}&=[x^{n+(d-1)r-1}]P_1^{r(d-1)}P_1^{r_1}P_2^{r_2}\cdots P_{k-i+1}^{r_{k-i+1}}\\ &=[x^{n+(d-1)r-1}]((\sqrt[d]{x})^{d-1}\sqrt[d]{y})^{r(d-1)}((\sqrt[d]{x})^{d-1}\sqrt[d]{y})^{r_1}\cdot\left((\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)\right)^{r_2}\\ &\cdots\left((\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{k-i}\right)^{r_{k-i+1}}\\ &=[x^{n+(d-1)r-1}]x^{r(d-1)}y^r(1-y)^{r_2}(1-y)^{2r_3}\cdots(1-y)^{(k-i)r_{k-i+1}}\\ &=[x^{n-1}]y^r(1-y)^s\end{aligned}$$

where $y = z(1-y)^{-kd}$ and $s := r_2 + 2r_3 + \cdots + (k-i)r_{k-i+1}$. Applying Lagrange-Bürmann inversion, we obtain

$$\begin{aligned}
&[x^n]x^{(1-d)r+1}P_1^{r(d-1)}P_1^{r_1}P_2^{r_2}\cdots P_{k-i+1}^{r_{k-i+1}}\\
&\qquad=\frac{1}{n-1}[y^{n-2}](ry^{r-1}(1-y)^s - sy^r(1-y)^{s-1})(1-y)^{-dk(n-1)}\\
&\qquad=\frac{1}{n-1}(r[y^{n-r-1}](1-y)^{-dk(n-1)+s} - s[y^{n-r-2}](1-y)^{-dk(n-1)+s-1}).
\end{aligned}$$

Making use of binomial theorem, we get

$$\begin{aligned}
&[x^n]x^{(1-d)r+1}P_1^{r(d-1)}P_1^{r_1}P_2^{r_2}\cdots P_{k-i+1}^{r_{k-i+1}}\\
&\quad=\frac{1}{n-1}\left[r[y^{n-r-1}]\sum_{a\geq 0}\binom{dk(n-1)-s+a-1}{a}y^a - s[y^{n-r-2}]\sum_{a\geq 0}\binom{dk(n-1)-s+a}{a}y^a\right]\\
&\quad=\frac{1}{n-1}\left[r\binom{(dk+1)(n-1)-s-r-1}{n-r-1} - s\binom{(dk+1)(n-1)-s-r-1}{n-r-2}\right]\\
&\quad=\frac{dkr-s}{(dk+1)(n-1)-s-r}\binom{(dk+1)(n-1)-s-r}{n-r-1}.
\end{aligned}$$

There are

$$\binom{r}{r_1, r_2, r_3, \ldots, r_{k-i+1}}$$

ways of assigning labels to the children of the root so that there are r_j children labeled by j for $j = 1, 2, \ldots, k-i+1$. The proof thus follows. □

We note that formula (10) follows from (16), by letting $s = r(i-1)$ and $r_j = 0$ for all $j \neq i$. If $s = r$ in Theorem 3.5 then it implies that $r_2 = r$, $r_1 = r_3 = r_4 = \cdots = r_{k-i+1} = 0$. This means that there are

$$\frac{(dk-1)r}{(dk+1)(n-1)-2r}\binom{(dk+1)(n-1)-2r}{n-r-1}$$

d-dimensional k-plane trees on n vertices such that the root of degree r is labeled by j and all the children of the root are labeled 2.

If $k = 2$ and $i = 1$ in (16) then $r_1 + r_2 = r$ and $r_2 = s$. This means that $r_2 = r - r_1$ and $s = r - r_1$. It then follows that that there are

$$\frac{(dk-1)r+r_1}{(dk+1)(n-1)-2r+r_1}\binom{(dk+1)(n-1)-2r+r_1}{n-r-1}\binom{r}{r_1, r-r_1} \tag{17}$$

d-dimensional 2-plane trees on n vertices with root labeled 1 and has r children of which r_1 are labeled 1. Summing over all values of r_1 and r in (17), we find the total number of d-dimensional 2-plane trees on n vertices with root labeled 1.

If $k = 2$ and $i = 2$ in (16) then $r_1 = r$ and $s = 0$. It implies that there

$$\frac{r}{n-1}\binom{(2d+1)(n-1)-r-1}{n-r-1}$$

d-dimensional 2-plane trees on n vertices with root labeled 2 and has r children all labeled 1.

4 Forests

In this section, we enumerate d-dimensional k-plane forests whose vertices are labeled from a given set. The vertices of the forest are labeled with integers $1,2,\ldots,n$ if the number of vertices is n.

Theorem 4.1. *There are*

$$\frac{n!(dkc-s)}{(dk+1)n-s-c}\binom{(dk+1)n-s-c}{n-c}\binom{c}{c_1,c_2,\ldots,c_{k-i+1}} \tag{18}$$

labeled d-dimensional k-plane forests on n vertices such that there are c components, c_j of which have roots labeled j where $j=1,2,\ldots,k-i+1$ and $s := c_2+2c_3+\cdots+(k-i)c_{k-i+1}$.

Proof. Let $P_i(x)$ be the generating function for d-dimensional k-plane trees rooted at a vertex labeled by i, where x marks a vertex. Since in the forest, there are r_i trees labeled $i=1,2,\ldots,k$, the generating function for unlabeled d-dimensional k-plane forest in which the positions of individual trees are not taken into consideration is

$$\begin{aligned}\left(\frac{P_1(x)^d}{x^{d-1}}\right)^{c_1}\left(\frac{P_1(x)^{d-1}P_2(x)}{x^{d-1}}\right)^{c_2}\cdots\left(\frac{P_1(x)^{d-1}P_{k-i+1}(x)}{x^{d-1}}\right)^{c_{k-i+1}}\\ = x^{(1-d)c}P_1^{c(d-1)}P_1^{c_1}P_2^{c_2}\cdots P_{k-i+1}^{c_{k-i+1}}.\end{aligned}$$

We extract the coefficient x^n in the generating function.

$$\begin{aligned}[x^n]x^{(1-d)c}P_1^{c(d-1)}P_1^{c_1}P_2^{c_2}\cdots P_{k-i+1}^{c_{k-i+1}} &= [x^{n+(d-1)c}]P_1^{c(d-1)}P_1^{c_1}P_2^{c_2}\cdots P_{k-i+1}^{c_{k-i+1}}\\ &= [x^{n+(d-1)c}]((\sqrt[d]{x})^{d-1}\sqrt[d]{y})^{c(d-1)}((\sqrt[d]{x})^{d-1}\sqrt[d]{y})^{c_1}\cdot\Big((\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)\Big)^{c_2}\\ &\cdots\Big((\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{k-i}\Big)^{c_{k-i+1}}\\ &= [x^{n+(d-1)c}]x^{c(d-1)}y^r(1-y)^{c_2}(1-y)^{2c_3}\cdots(1-y)^{(k-i)c_{k-i+1}}\\ &= [x^n]y^c(1-y)^s.\end{aligned}$$

Here, $y=z(1-y)^{-kd}$ and $s := c_2+2c_3+\cdots+(k-i)c_{k-i+1}$.

By Lagrange-Bürmann inversion, we have

$$\begin{aligned}[x^n]x^{(1-d)c}P_1^{c(d-1)}P_1^{c_1}P_2^{c_2}\cdots P_{k-i+1}^{c_{k-i+1}} &= \frac{1}{n}[y^{n-1}](ry^{c-1}(1-y)^s - sy^c(1-y)^{s-1})(1-y)^{-dkn}\\ &= \frac{1}{n}(c[y^{n-c}](1-y)^{-dkn+s} - s[y^{n-c-1}](1-y)^{-dkn+s-1}).\end{aligned}$$

Binomial theorem gives,

$$\begin{aligned}
&[x^n]x^{(1-d)c}P_1^{c(d-1)}P_1^{c_1}P_2^{c_2}\cdots P_{k-i+1}^{c_{k-i+1}} \\
&\qquad = \frac{1}{n}\left[c[y^{n-r}]\sum_{a\geq 0}\binom{dkn-s+a-1}{a}y^a - s[y^{n-c-1}]\sum_{a\geq 0}\binom{dkn-s+a}{a}y^a\right] \\
&\qquad = \frac{1}{n}\left[c\binom{(dk+1)n-s-c-1}{n-c} - s\binom{(dk+1)n-s-c-1}{n-c-1}\right] \\
&\qquad = \frac{dkc-s}{(dk+1)n-s-c}\binom{(dk+1)n-s-c}{n-c}.
\end{aligned}$$

Since there are

$$\binom{c}{c_1,c_2,\ldots,c_{k-i+1}}$$

choices for positions of the trees in the forest and $n!$ choices for labeling of the vertices in the tree then result follows by product rule of counting. □

If $c_i = c$ in (18) then $s = c(i-1)$ and $c_j = 0$ for all $j \neq i$. This implies that

$$\frac{n!(dk-i+1)c}{(dk+1)n-c(i-2)}\binom{(dk+1)n-c(i-2)}{n-c} \tag{19}$$

counts labeled d-dimensional k-plane forests with n vertices and c components such that the roots of all the trees are labeled by i.

Corollary 4.2. *There are*

$$\frac{n!(kc-s)}{(k+1)n-s-c}\binom{(k+1)n-s-c}{n-c}\binom{c}{c_1,c_2,\ldots,c_{k-i+1}}$$

labeled k-plane forests on n vertices such that there are c components, c_j of which have roots labeled by j where $j = 1,2,\ldots,k-i+1$ and $s := c_2 + 2c_3 + \cdots + (k-i)c_{k-i+1}$.

Proof. Set $d = 1$ in (18). □

Upon setting $d = 2$ in (18), we obtain the following corollary.

Corollary 4.3. *There are*

$$\frac{n!(2kc-s)}{(2k+1)n-s-c}\binom{(2k+1)n-s-c}{n-c}\binom{c}{c_1,c_2,\ldots,c_{k-i+1}}$$

labeled k-noncrossing forests on n vertices and c components, c_j of which have root labeled by j where $j = 1,2,\ldots,k-i+1$ and $s := c_2 + 2c_3 + \cdots + (k-i)c_{k-i+1}$.

5 Eldest child of the root

In the section, we enumerate the set of generalized k-plane trees by the label of the root as well as the label of the eldest child of the root.

Theorem 5.1. *The number of d-dimensional k-plane trees on n vertices with root labeled by i such that the eldest child of the root is labeled by j is given by*

$$\frac{dk+k-i-j+2}{(dk+1)n-k(d-1)-i-j}\binom{(dk+1)n-k(d-1)-i-j}{n-2}. \tag{20}$$

Proof. Let $P_i(x)$ be the generating function for d-dimensional k-plane trees rooted at a vertex labeled by i, where x marks a vertex. The generating function for d-dimensional k-plane trees rooted a vertex labeled by i such that the first child of the root is labeled by j where $i+j \leq k+1$ is thus

$$P_i(x) \cdot \frac{P_1(x)^{d-1}P_j(x)}{x^{d-1}}.$$

The desired result is obtained by extracting the coefficient of x^n in the generating function.

$$\begin{aligned}
[x^n]P_i(x) \cdot \frac{P_1(x)^{d-1}P_j(x)}{x^{d-1}} &= [x^n](\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{i-1} \cdot \frac{((\sqrt[d]{x})^{d-1}\sqrt[d]{y})^{d-1}(\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{j-1}}{x^{d-1}} \\
&= [x^n]x^{(d-1)/d}y^{(d+1)/d}(1-y)^{i+j-2} \\
&= [x^{n-1+1/d}]y^{1+1/d}(1-y)^{i+j-2}.
\end{aligned}$$

As before, y satisfies the functional equation $y = z(1-y)^{-kd}$.

Lagrange-Bürmann inversion gives,

$$\begin{aligned}
&[x^n]P_i(x)\cdot\frac{P_1(x)^{d-1}P_j(x)}{x^{d-1}} \\
&\quad=\frac{1}{n-1+1/d}[y^{n-2+1/d}]\left(\frac{d+1}{d}y^{1/d}(1-y)^{i+j-2}-(i+j-2)y^{1+1/d}(1-y)^{i+j-3}\right) \\
&\quad\quad(1-y)^{-dk(n-1+1/d)} \\
&\quad=\frac{1}{d(n-1)+1}[y^{n-2}]\,(d+1-(1+d(i+j-1))y) \\
&\quad\quad(1-y)^{-(k(d(n-1)+1)-i-j+3)}.
\end{aligned}$$

We use binomial theorem to get,

$$\begin{aligned}
[x^n]P_i(x)\cdot\frac{P_1(x)^{d-1}P_j(x)}{x^{d-1}} =&\frac{1}{d(n-1)+1}[y^{n-2}]\,(d+1-(1+d(i+j-1))y)\\
&\sum_{a\geq 0}\binom{k(d(n-1)+1)-i-j+a+2}{a}y^a\\
=&\frac{1}{d(n-1)+1}\left[(d+1)\binom{(dk+1)n-k(d-1)-i-j}{n-2}\right.\\
&\left.-(1+d(i+j-1))\binom{(dk+1)n-k(d-1)-i-j-1}{n-3}\right]\\
=&\frac{dk+k-i-j+2}{(dk+1)n-k(d-1)-i-j}\binom{(dk+1)n-k(d-1)-i-j}{n-2}.
\end{aligned}$$

This completes the proof. □

Setting $i+j=k+1$ in (20), we find that the number of d-dimensional k-plane trees on n vertices such that the sum of the labels of the root and its eldest child is $k+1$ is

$$\frac{1}{n-1}\binom{(dk+1)(n-1)}{n-2}.$$

The same formula counts d-dimensional k-plane trees on n vertices with root labeled by k since in such trees all the children of the root are labeled 1, i.e., the sum of the label of the root and its eldest child is $k+1$. The formula therefore holds if $i=k$ and $j=1$ in (20).

We obtain the the following result upon setting $d=1$ in (20).

Corollary 5.2. *There are*

$$\frac{2k-i-j+2}{(k+1)n-i-j}\binom{(k+1)n-i-j}{n-2}$$

k-plane trees on n vertices such that the root is labeled by i and the eldest child of the root is labeled by j.

On letting $d=2$ in (20), we obtain the following corollary.

Corollary 5.3. *The number of k-noncrossing trees on n vertices such that the root is labeled by i and the eldest child of the root is labeled by j is given by*

$$\frac{3k-i-j+2}{(2k+1)n-k-i-j}\binom{(2k+1)n-k-i-j}{n-2}.$$

If $k=1$, then $i=1$ and $j=1$ and thus substituting these values in (20), we obtain

$$\frac{1}{n-1}\binom{(d+1)(n-1)}{n-2}$$

as the formula for the number of d-dimensional plane trees on n vertices. This formula was obtained by Okoth and Kasyoki in [25].

6 Leftmost path

The following section is set aside for enumeration of d-dimensional k-plane trees by length of the leftmost path.

Theorem 6.1. *The number of d-dimensional k-plane trees on n vertices whose root is labeled by i such that there is a leftmost path of length $\ell \geq 0$ and all other vertices on the path are labeled by j where $i + j \leq k + 1$ is given by*

$$\frac{\ell(dk-j+1)+k-i+1}{(dk+1)n-k(d-1)-j\ell-i}\binom{(dk+1)n-k(d-1)-j\ell-i}{n-\ell-1}. \tag{21}$$

Proof. The generating function for the d-dimensional k-plane trees described in the statement of the theorem is $\left(\frac{P_1(x)^{d-1}P_j(x)}{x^{d-1}}\right)^{\ell} P_i(x)$. We extract the coefficient of x^n in the generating function as follows.

$$\begin{aligned}
&[x^n]\left(\frac{P_1(x)^{d-1}P_j(x)}{x^{d-1}}\right)^{\ell} \cdot P_i(x)\\
&= [x^n]\left(\frac{((\sqrt[d]{x})^{d-1}\sqrt[d]{y})^{d-1}(\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{j-1}}{x^{d-1}}\right)^{\ell} \cdot (\sqrt[d]{x})^{d-1}\sqrt[d]{y}(1-y)^{i-1}\\
&= [x^n]x^{1-1/d}y^{\ell+1/d}(1-y)^{\ell(j-1)+i-1}\\
&= [x^{n-1+1/d}]y^{\ell+1/d}(1-y)^{\ell(j-1)+i-1}.
\end{aligned}$$

We then use Lagrange Bürmann inversion to obtain

$$\begin{aligned}
&[x^n]\left(\frac{P_1(x)^{d-1}P_i(x)}{x^{d-1}}\right)^{\ell} \cdot P_i(x)\\
&\qquad = \frac{1}{n-1+1/d}[y^{n-2+1/d}]\left[\frac{d\ell+1}{d}y^{\ell-1+1/d}(1-y)^{\ell(j-1)+i-1}\right.\\
&\qquad\quad \left.-(\ell(j-1)+i-1)(1-y)^{\ell(j-1)+i-2}y^{\ell+1/d}\right](1-y)^{-dk(n-1+1/d)}\\
&\qquad = \frac{1}{d(n-1)+1}[y^{n-\ell-1}]\left((d\ell+1)(1-y)-dy(\ell(j-1)+i-1)\right)\\
&\qquad\quad (1-y)^{-(k(d(n-1)+1)-\ell(j-1)-i+2)}\\
&\qquad = \frac{1}{d(n-1)+1}[y^{n-\ell-1}]\left(d\ell+1-(d(j\ell+i-1)+1)y\right)\\
&\qquad\quad (1-y)^{-(k(d(n-1)+1)-\ell(j-1)-i+2)}.
\end{aligned}$$

Application of binomial theorem gives

$$
\begin{aligned}
&[x^n]\left(\frac{P_1(x)^{d-1}P_i(x)}{x^{d-1}}\right)^{\ell}\cdot P_i(x)\\
&\qquad=\frac{1}{d(n-1)+1}[y^{n-\ell-1}]\,(d\ell+1-(d(j\ell+i-1)+1)y)\\
&\qquad\quad\sum_{a\geq 0}\binom{k(d(n-1)+1)-\ell(j-1)-i+a+1}{a}y^a\\
&\qquad=\frac{1}{d(n-1)+1}\left[(d\ell+1)\binom{(dk+1)(n-1)+k-j\ell-i+1}{n-\ell-1}\right.\\
&\qquad\quad\left.-(d(j\ell+i-1)+1)\binom{(dk+1)(n-1)+k-j\ell-i}{n-\ell-2}\right]\\
&\qquad=\frac{dk\ell+k+\ell-j\ell-i+1}{(dk+1)(n-1)+k-j\ell-i+1}\binom{(dk+1)(n-1)+k-j\ell-i+1}{n-\ell-1}.
\end{aligned}
$$

This completes the proof. □

On setting $d=1$ in (21), we get the following result.

Corollary 6.2. *The number of k-plane trees on n vertices with root labeled by i such that there is a leftmost path of length $\ell \geq 0$ and all other vertices on the path are labeled by j is given by*

$$\frac{\ell(k-j+1)+k-i+1}{(k+1)n-j\ell-i}\binom{(k+1)n-j\ell-i}{n-\ell-1}.$$

Also, letting $d=2$ in (21), we obtain:

Corollary 6.3. *The number of k-noncrossing trees on n vertices with root labeled by i such that there is a leftmost path of length $\ell \geq 0$ and all other vertices on the path are labeled by j is given by*

$$\frac{\ell(2k-j+1)+k-i+1}{(2k+1)n-k-j\ell-i}\binom{(2k+1)n-k-j\ell-i}{n-\ell-1}.$$

If $\ell=1$ in (21), we get the following corollary.

Corollary 6.4. *The number of d-dimensional k-plane trees on n vertices with root labeled by i such that the eldest child of the root is labeled by j is given by*

$$\frac{dk+k-j-i+2}{(dk+1)n-k(d-1)-j-i}\binom{(dk+1)n-k(d-1)-j-i}{n-2}. \tag{22}$$

Formula (22) was also obtained in (20). If we set $\ell=0$ in (21), then we get

$$\frac{k-i+1}{(dk+1)n-k(d-1)-i}\binom{(dk+1)n-k(d-1)-i}{n-1} \tag{23}$$

as the formula for d-dimensional k-plane trees on n vertices such that the root is labeled by i.

On further setting $d=1$ and $d=2$ in (23), we rediscover the following corollaries.

Corollary 6.5. *The number of k-plane trees on n vertices with root labeled by i is*

$$\frac{k-i+1}{(k+1)n-i}\binom{(k+1)n-i}{n-1}.$$

Corollary 6.6. *The number of k-noncrossing trees on n vertices with root labeled by i is*

$$\frac{k-i+1}{(2k+1)n-k-i}\binom{(2k+1)n-k-i}{n-1}.$$

These results were obtained by Okoth and Wagner in [26].

Corollary 6.7. *The number of d-dimensional k-plane trees on n vertices whose root is labeled by i such that there is a leftmost path of length $\ell \geq 0$ and all other vertices on the path are labeled by i where $2i \leq k+1$ is given by*

$$\frac{k(d\ell+1)-(i-1)(\ell+1)}{(dk+1)n-k(d-1)-i\ell-i}\binom{(dk+1)n-k(d-1)-i\ell-i}{n-\ell-1}. \tag{24}$$

Proof. Set $j = i$ in (21). □

Setting $i = 1$ in (24), we get that:

Corollary 6.8. *The number of d-dimensional k-plane trees on n vertices whose root is labeled 1 such that there is a leftmost path of length $\ell \geq 0$ and all other vertices on the path are labeled 1 is given by*

$$\frac{d\ell+1}{d(n-1)+1}\binom{(dk+1)n-k(d-1)-\ell-2}{n-\ell-1}. \tag{25}$$

Upon letting $d = 1$ and $d = 2$ in (25), we arrive at the following corollaries.

Corollary 6.9. *The number of k-plane trees on n vertices with root labeled 1 and a leftmost path of length $\ell \geq 0$ such that all vertices on the path are labeled 1 is*

$$\frac{\ell+1}{n}\binom{(k+1)n-\ell-2}{n-\ell-1}.$$

Corollary 6.10. *The number of k-noncrossing trees on n vertices with root labeled 1 and a leftmost path of length $\ell \geq 0$ such that all vertices on the path are labeled 1 is*

$$\frac{2\ell+1}{2n-1}\binom{(2k+1)n-k-\ell-2}{n-\ell-1}.$$

Now, setting $k = 1$ in (25), we find that there are

$$\frac{d\ell+1}{d(n-1)+1}\binom{(d+1)(n-1)-\ell}{n-\ell-1} \tag{26}$$

d-dimensional plane trees on n vertices with a leftmost path of length $\ell \geq 0$. With $\ell = 0$ in (26), it follows that there are

$$\frac{1}{d(n-1)+1}\binom{(d+1)(n-1)}{n-1} \tag{27}$$

d-dimensional plane trees on n vertices. This is a generalization of Catalan numbers which was also derived by Okoth and Kasyoki in [25].

7 Conclusion

The present study deals with the unification of the results already obtained for k-plane trees and k-noncrossing with respect to number of vertices, root degree, number of forests with a given number of components, label of the eldest child of the root and length of the leftmost paths. Statistic such that the occurrence of vertices of a given label type has not been tackled. The study can be extended to handle the statistic as well as parameters such as degree sequence, degree of a given vertex at a certain level, number of descents, number of leaves and many other statistics. Various bijections of k-plane trees and k-noncrossing trees have constructed [8,13,19]. Bijections of d-dimensional k-plane trees can also be investigated. Other than labeling vertices of plane trees and noncrossing in a certain way, these trees have been generalized by considering their block graphs [9, 20, 22, 27]. It would be interesting to consider and enumerate d-dimensional versions of these tree-like structures.

Acknowledgments. All the authors thank German Academic Exchange Service (DAAD) for sponsoring the first author in his PhD study at Moi University, Kenya. The second author thanks Maseno University for providing necessary resources to conduct this study. The authors acknowledge the valuable comments from the anonymous reviewers that improved clarity of concepts presented in this paper.

Funding

This research received no external funding.

Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declare that they have no competing interests.

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Citation: A. O. Nyariaro, I. O. Okoth, F. O. Nyamwala, Generalized k-plane trees, J. Disc. Math. Appl. 10(2) (2025) 161–182.
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