Journal of Discrete Mathematics and Its Applications 10 (2) (2025) 157–160

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Note

Proof of a conjecture on Sombor index

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Academic Editor: Ivan Gutman

Abstract. Let G be a simple graph. In this paper we prove the stated conjecture in [Some notes on Sombor index of graphs, MATCH Commun. Math. Comput. Chem.93 (2025) 853-859] and state an upper bound for the Sombor index of G in terms of its size and the smallest eigenvalue of its Sombor matrix. Also, characterize all graphs for which the specified bound is attained.

Keywords. Sombor index, Sombor matrix, eigenvalue.
Mathematics Subject Classification (2020): 05C09, 05C90.

1 Introduction

Let $G = (V(G), E(G))$ be a undirected simple graph, where $V(G)$ and $E(G)$ are the vertex and the edge sets of G, respectively. By the *order* and *size* of G, we mean the number of its vertices and edges. The degree d_u of a vertex $u \in V(G)$ is the number of edges that have u as an endpoint. The *open neighborhood* of u is the set $N(u) = \{v \in V(G) : uv \in E(G)\}$. We denote by K_2 the connected graph of order 2. Also, denote by $H_{m,n}$ the graph consisting of m copies of K_2 and $n - 2m$ isolated vertices.

According to Gutman [3], the *Sombor index* of G is defined as $SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$. Some results for the mentioned index can be found in [1,4–6,8,9] and the references therein. The *Sombor matrix* $A_{SO(G)} = (s_{ij})$ of the graph G is defined by [2] $s_{ij} = \sqrt{d_{v_i}^2 + d_{v_j}^2}$ if v_i and

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Received 28 March 2025; Revised 29 March 2025; Accepted 02 May 2025
First Publish Date: 01 June 2025

v_j are adjacent and 0 otherwise. We denote its eigenvalues by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Note that $\sigma_1(K_2) = \sigma_1(H_{m,n}) = \sqrt{2}$. In [7] the authors proved and proposed the following theorem and conjecture, respectively.

Theorem 1.1. *[7, Theorem 5] Let G be a graph of size $m > 0$ and order n. Then $m\sigma_1 \geq SO(G)$. Also, If G is connected, then equality holds if and only if $G \cong K_2$. In the general case, equality holds if and only if $G \cong H_{m,n}$, when $SO(G) = \sqrt{2}m$.*

Conjecture 1.2. *Let G be a graph of size m and order n. Then*

$$m\,|\sigma_n| = -m\,\sigma_n \geq SO(G)\,. \tag{1}$$

Moreover, if G is connected, then equality in (1) holds if and only if G is the complete graph. Then $\sigma_n = -\sqrt{2}(n-1)$ and $SO(G) = \sqrt{2}(n-1)\,\frac{n(n-1)}{2}$. In the general case, equality holds if and only if G consists of mutually isomorphic complete graphs and some (or no) isolated vertices.

2 Proof of the conjecture

In this paper, we prove the conjecture.

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$. First note that if $m = 0$, then $SO(G) = 0$ and hence the relation (1) holds. So, suppose that $m \geq 1$ and let $e = v_iv_j$ be an arbitrary edge in $E(G)$. With out loss of generality suppose $i < j$. Consider

$$X_{i,j} = (0, \ldots, 0, -1, 0, \ldots, 0, 1, 0 \ldots, 0)^T$$

as a n-tuple with entries in $\{-1, 0, 1\}$ such that the i^{th} and the j^{th} entries of it are equal to -1 and 1, respectively. By the Rayleigh–Ritz variational principle, we obtain:

$$\sigma_n \leq \frac{X_{i,j}^T A_{SO(G)} X_{i,j}}{X_{i,j}^T X_{i,j}}.$$

It conclude that $\sigma_n \leq -\dfrac{s_{ij} + s_{ji}}{2}$ and consequently $\sqrt{d_i^2 + d_j^2} \leq -\sigma_n$. This implies that

$$\textstyle\sum_{e=v_iv_j \in E(G)} \sqrt{d_i^2 + d_j^2} \leq -m\sigma_n$$

and so $SO(G) \leq -m\sigma_n$. Moreover, equality in (1) will hold if and only if $X_{i,j}$ is the eigenvector of $A_{SO(G)}$ corresponding to the eigenvalue σ_n, for all i, j with $v_iv_j \in E(G)$. It concludes that

$$N(v_i) \setminus \{v_j\} = N(v_j) \setminus \{v_i\} \tag{2}$$

Let G' be the connected component of G such that $e = v_iv_j \in E(G')$. We claim that G' is a complete subgraph of G. To the contrary, suppose $G'' \neq G'$ is a maximal clique of G' and $x \in V(G') \setminus V(G'')$. Since G' is connected, there exists a vertex $y \in V(G'')$ such that $x \in N(y)$. This implies that $x \in N(z)$ for any $z \in V(G'')$, by (2), a contradiction. Until now, we prove that each connected component of G is complete. Next, applying $\sqrt{d_i^2 + d_j^2} = -\sigma_n$ implies that every connected component of G has the same order and the proof is complete. □

3 Conclusions

For an arbitrary simple undirected graph G of size m, we proved that $SO(G) \leq -m\sigma_n$, where σ_n is the smallest eigenvalue of the Sombor matrix $A_{SO(G)}$. Moreover it is shown that equality holds if and only if G consists of mutually isomorphic complete graphs and some (or no) isolated vertices.

Acknowledgments

The authors would like to express their gratitude to the referee for her/his careful reading and helpful comments.

Funding

This research received no external funding.

Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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Citation: M. Habibi, R. Singh, Proof of a Conjecture on Sombor Index, J. Disc. Math. Appl. 10(2) (2025) 157–160.
https://doi.org/10.22061/jdma.2025.11904.1126