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Research Paper

Elliptic Sombor energy of a graph

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Abstract. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The elliptic Sombor matrix of G, denoted by $A_{ESO}(G)$, is defined as the $n \times n$ matrix whose (i,j)-entry is $(d_i + d_j)\sqrt{d_i^2 + d_j^2}$ if v_i and v_j are adjacent and 0 for another cases. Let the eigenvalues of the elliptic Sombor matrix $A_{ESO}(G)$ be $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$ which are the roots of the elliptic Sombor characteristic polynomial $\prod_{i=1}^{n}(\rho - \rho_i)$. The elliptic Sombor energy E_{ESO} of G is the sum of absolute values of the eigenvalues of $A_{ESO}(G)$. In this paper, we compute the elliptic Sombor characteristic polynomial and the elliptic Sombor energy for some graph classes. We compute the elliptic Sombor energy of cubic graphs of order 10 and as a consequence, we see that two k-regular graphs of the same order may have different elliptic Sombor energy.

Keywords. elliptic Sombor matrix, elliptic Sombor energy, elliptic Sombor characteristic polynomial, eigenvalues, regular graphs.
Mathematics Subject Classification (2020): 05C09, 05C90.

1 Introduction

Let $G = (V, E)$ be a simple graph, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. If two vertices v_i and v_j of G are adjacent, then we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by d_i, is the number of the vertices adjacent to v_i.

Let $A(G)$ be adjacency matrix of G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ its eigenvalues. These are said to be

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the eigenvalues of the graph G and to form its spectrum [9]. The energy $E(G)$ of the graph G is defined as the sum of the absolute values of its eigenvalues

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Details and more information on graph energy can be found in [21,22,24,29]. There are many kinds of graph energies, such as Randić energy [1,3,5,12,23], distance energy [33], incidence energy [4], matching energy [7,27] and Laplacian energy [11].

Sombor index is defined as $SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$ (see [19]). More details on Sombor index can be found in [2,6,8,10,13,17,28,30–32,34]. Recently, in [20], Gutman introduced Sombor matrix of a graph G as $A_{SO}(G) = (r_{ij})_{n \times n}$, where

$$r_{ij} = \begin{cases} \sqrt{d_i^2 + d_j^2} & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A_{SO}(G)$ are denoted by $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$, and are said to form the Sombor spectrum of the graph G. The Sombor characteristic polynomial $\phi_{SO}(G, \lambda)$ is

$$\phi_{SO}(G, \lambda) = det(\lambda I - A_{SO}(G)) = \prod_{i=1}^{n} (\lambda - \rho_i),$$

and Sombor energy $E_{SO}(G)$ is

$$E_{SO}(G) = \sum_{i=1}^{n} |\rho_i|.$$

We refer the reader to [16,18,25,26] for more details on Sombor energy.

In a recent paper (see [14]), the elliptic Sombor index of G is defined as

$$ESO(G) = \sum_{uv \in E(G)} (d_u + d_v)\sqrt{d_u^2 + d_v^2}.$$

Motivated by the definition of the Sombor matrix, we define the elliptic Sombor matrix as $A_{ESO}(G) = (r_{ij})_{n \times n}$, and

$$r_{ij} = \begin{cases} (d_i + d_j)\sqrt{d_i^2 + d_j^2} & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A_{ESO}(G)$ are denoted by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and are said to form the elliptic Sombor spectrum of the graph G. The elliptic Sombor characteristic polynomial $\phi_{ESO}(G, \lambda)$ is

$$\phi_{ESO}(G, \lambda) = det(\lambda I - A_{ESO}(G)) = \prod_{i=1}^{n} (\lambda - \lambda_i),$$

and elliptic Sombor energy $E_{ESO}(G)$ is

$$E_{ESO}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Two graphs G and H are said to be *elliptic Sombor energy equivalent*, or simply $\mathcal{E}_{\mathcal{ESO}}$-equivalent, written $G \sim H$, if $E_{ESO}(G) = E_{ESO}(H)$. It is evident that the relation $\sim$ of being $\mathcal{E}_{\mathcal{ESO}}$-equivalence is an equivalence relation on the family $\mathcal{G}$ of graphs, and thus $\mathcal{G}$ is partitioned into equivalence classes, called the $\mathcal{E}_{\mathcal{ESO}}$*-equivalent*. Given $G \in \mathcal{G}$, let

$$[G] = \{H \in \mathcal{G} : H \sim G\}.$$

We call $[G]$ the equivalence class determined by G. A graph G is said to be *elliptic Sombor energy unique*, or simply $\mathcal{E}_{\mathcal{ESO}}$*-unique*, if $[G] = \{G\}$.

A graph G is called *k-regular* if all vertices have the same degree k. One of the famous graphs is the Petersen graph which is a symmetric non-planar 3-regular graph of order 10.

There are exactly twenty one 3-regular graphs of order 10 [15]. In the study of elliptic Sombor energy, it is natural to investigate the elliptic Sombor characteristic polynomial and elliptic Sombor energy of cubic graphs of order 10 and check whether they recognized by their elliptic Sombor energy among other 3-regular graphs with the same order. We denote the Petersen graph by P.

In the next section we compute the elliptic Sombor energy of specific graphs. In Section 3, we study the elliptic Sombor energy of cubic graphs of order 10. As a consequence we show that the Petersen graph cannot be determined by its elliptic Sombor energy.

2 Elliptic Sombor energy of specific graphs

In this section, we study the elliptic Sombor characteristic polynomial and the elliptic Sombor energy for certain graphs. Here we compute the elliptic Sombor characteristic polynomial of paths and cycles.

Theorem 2.1. *For every $n \geq 5$, the elliptic Sombor characteristic polynomial of the path graph P_n satisfy:*

$$\phi_{ESO}(P_n, \lambda) = \lambda^2 \Lambda_{n-2} - 90\lambda \Lambda_{n-3} + 2025 \Lambda_{n-4},$$

where for every $k \geq 3$, $\Lambda_k = \lambda \Lambda_{k-1} - 128 \Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - 8$. Also the characteristic polynomial of P_2, P_3 and P_4 are $\lambda^2 - 8, \lambda^3 - 90\lambda$ and $\lambda^4 - 218\lambda^2 + 2025$ respectively.

Proof. It is easy to see that the characteristic polynomial of P_2 is $\lambda^2 - 8$, Also for P_3 is $\lambda^3 - 90\lambda$ and for P_4 is $\lambda^4 - 218\lambda^2 + 2025$. Now for every $k \geq 3$ consider

$$M_k := \begin{pmatrix} \lambda & -4\sqrt{8} & 0 & \dots & 0 & 0 \\ -4\sqrt{8} & \lambda & -4\sqrt{8} & \dots & 0 & 0 \\ 0 & -4\sqrt{8} & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & -4\sqrt{8} \\ 0 & 0 & 0 & \dots & -4\sqrt{8} & \lambda \end{pmatrix}_{k \times k},$$

and let $\Lambda_k = det(M_k)$. One can easily check that $\Lambda_k = \lambda\Lambda_{k-1} - 128\Lambda_{k-2}$. Now consider the path graph P_n. Suppose that $\phi_{ESO}(P_n,\lambda) = det(\lambda I - A_{ESO}(P_n))$. We have

$$\phi_{ESO}(P_n,\lambda) = det\left(\begin{array}{c|cccccc|c} \lambda & -3\sqrt{5} & 0 & 0 & \dots & 0 & 0 & 0 \\ \hline -3\sqrt{5} & & & & & & & 0 \\ 0 & & & & & & & 0 \\ 0 & & & & & & & 0 \\ \vdots & & & & M_{n-2} & & & \vdots \\ 0 & & & & & & & 0 \\ 0 & & & & & & & -3\sqrt{5} \\ \hline 0 & 0 & 0 & 0 & \dots & 0 & -3\sqrt{5} & \lambda \end{array}\right)_{n\times n}.$$

So,

$$\phi_{ESO}(P_n,\lambda) = \lambda det\left(\begin{array}{ccc|c} & & & 0 \\ & M_{n-2} & & \vdots \\ & & & 0 \\ & & & -3\sqrt{5} \\ \hline 0\dots & 0 & -3\sqrt{5} & \lambda \end{array}\right)$$

$$+ 3\sqrt{5}det\left(\begin{array}{c|ccc|c} -3\sqrt{5} & -4\sqrt{8} & \dots & 0 & 0 \\ \hline 0 & & & & 0 \\ \vdots & & M_{n-3} & & \vdots \\ 0 & & & & -3\sqrt{5} \\ \hline 0 & 0 & \dots & -3\sqrt{5} & \lambda \end{array}\right).$$

And so,

$$\phi_{ESO}(P_n,\lambda) = \lambda\left(\lambda\Lambda_{n-2} + 3\sqrt{5}det\left(\begin{array}{ccc|c} & & & 0 \\ & M_{n-3} & & \vdots \\ & & & 0 \\ & & & 0 \\ \hline 0\dots & 0 & -4\sqrt{8} & -3\sqrt{5} \end{array}\right)\right)$$

$$- 45det\left(\begin{array}{ccc|c} & & & 0 \\ & M_{n-3} & & \vdots \\ & & & 0 \\ & & & -3\sqrt{5} \\ \hline 0\dots & 0 & -3\sqrt{5} & \lambda \end{array}\right)$$

Hence,

$$\begin{aligned}\phi_{ESO}(P_n,\lambda) &= \lambda\left(\lambda\Lambda_{n-2} - 45\Lambda_{n-3}\right)\\ &\quad - 45\left(\lambda\Lambda_{n-3} + 3\sqrt{5}det\left(\begin{array}{ccc|c} & & & 0\\ & M_{n-4} & & \vdots\\ & & & 0\\ & & & 0\\ \hline 0\ldots & 0 & -4\sqrt{8} & -3\sqrt{5}\end{array}\right)\right)\\ &= \lambda\left(\lambda\Lambda_{n-2} - 45\Lambda_{n-3}\right) - 45\left(\lambda\Lambda_{n-3} - 45\Lambda_{n-4}\right),\end{aligned}$$

and therefore we have the result. □

Theorem 2.2. *For every $n \geq 3$, the elliptic Sombor characteristic polynomial of the cycle graph C_n satisfy:*

$$\phi_{ESO}(C_n,\lambda) = \lambda\Lambda_{n-1} - 1024\Lambda_{n-2} + ((-1)^{n+1})(2)(8\sqrt{8})^n,$$

where for every $k \geq 3$, $\Lambda_k = \lambda\Lambda_{k-1} - 8\Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - 512$.

Proof. Similar to the proof of Theorem 2.1, for every $k \geq 3$, we consider

$$M_k := \begin{pmatrix} \lambda & -8\sqrt{8} & 0 & 0 & \ldots & 0 & 0 & 0\\ -8\sqrt{8} & \lambda & -8\sqrt{8} & 0 & \ldots & 0 & 0 & 0\\ 0 & -8\sqrt{8} & \lambda & -8\sqrt{8} & \ldots & 0 & 0 & 0\\ 0 & 0 & -8\sqrt{8} & \lambda & \ldots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & \lambda & -8\sqrt{8} & 0\\ 0 & 0 & 0 & 0 & \ldots & -8\sqrt{8} & \lambda & -8\sqrt{8}\\ 0 & 0 & 0 & 0 & \ldots & 0 & -8\sqrt{8} & \lambda \end{pmatrix}_{k\times k},$$

and let $\Lambda_k = det(M_k)$. We have $\Lambda_k = \lambda\Lambda_{k-1} - 512\Lambda_{k-2}$. Suppose that $\phi_{ESO}(C_n,\lambda) = det(\lambda I - A_{ESO}(C_n))$. We have

$$\phi_{ESO}(C_n,\lambda) = det\left(\begin{array}{c|c} \lambda & -8\sqrt{8}\ 0\ 0\ \ \ldots\ \ 0\ 0\ -8\sqrt{8}\\ \hline -8\sqrt{8} & \\ 0 & \\ 0 & \\ \vdots & M_{n-1}\\ 0 & \\ 0 & \\ -8\sqrt{8} & \end{array}\right)_{n\times n}.$$

So,

$$\phi_{ESO}(P_n,\lambda)=\lambda\Lambda_{n-1}+8\sqrt{8}det\left(\begin{array}{c|ccc} -8\sqrt{8} & -8\sqrt{8} & 0 & \dots 0\\ \hline 0 & & & \\ \vdots & & M_{n-2} & \\ 0 & & & \\ -8\sqrt{8} & & & \end{array}\right)$$

$$+(-1)^{n+1}(-8\sqrt{8})det\left(\begin{array}{c|ccc} -8\sqrt{8} & & & \\ 0 & & M_{n-2} & \\ \vdots & & & \\ 0 & & & \\ \hline -8\sqrt{8} & 0\dots & 0 & -8\sqrt{8} \end{array}\right).$$

Hence,

$$\begin{aligned}\phi_{ESO}(C_n,\lambda)&=\lambda\Lambda_{n-1}+8\sqrt{8}\left(-8\sqrt{8}\Lambda_{n-2}+(-1)^n(-8\sqrt{8})^{n-1}\right)\\ &\quad+(-1)^{n+1}(-8\sqrt{8})\left((-8\sqrt{8})^{n-1}+(-1)^n(-8\sqrt{8})\Lambda_{n-2}\right),\end{aligned}$$

and therefore we have the result. □

Now we consider to star graph S_n and find its elliptic Sombor characteristic polynomial and elliptic Sombor energy. We need the following Lemma.

Lemma 2.3. [9] If M is a nonsingular square matrix, then

$$det\begin{pmatrix} M & N\\ P & Q\end{pmatrix}=det(M)det(Q-PM^{-1}N).$$

Theorem 2.4. *For $n\geq 2$,*

(i) The elliptic Sombor characteristic polynomial of the star graph $S_n=K_{1,n-1}$ is

$$\phi_{ESO}(S_n,\lambda))=\lambda^{n-2}\left(\lambda^2-(n-1)(n^4-2n^3+2n^2)\right).$$

(ii) The elliptic Sombor energy of S_n is

$$E_{ESO}(S_n)=2n\sqrt{(n-1)(n^2-2n+2)}.$$

Proof. (i) One can easily check that the elliptic Sombor matrix of $K_{1,n-1}$ is

$$A_{ESO}(S_n)=n\sqrt{n^2-2n+2}\begin{pmatrix} 0_{1\times 1} & J_{1\times n-1}\\ J_{n-1\times 1} & 0_{n-1\times n-1}\end{pmatrix}.$$

We have $det(\lambda I - A_{ESO}(S_n)) =$

$$det \begin{pmatrix} \lambda & -n\sqrt{n^2-2n+2}J_{1\times(n-1)} \\ -n\sqrt{n^2-2n+2}J_{(n-1)\times 1} & \lambda I_{n-1} \end{pmatrix}.$$

Using Lemma 2.3, $det(\lambda I - A_{ESO}(S_n)) =$

$$\lambda det(\lambda I_{n-1} - n\sqrt{n^2-2n+2}J_{(n-1)\times 1}\frac{1}{\lambda}(n\sqrt{n^2-2n+2}J_{1\times(n-1)})).$$

Since $J_{(n-1)\times 1}J_{1\times(n-1)} = J_{n-1}$, so

$$\begin{aligned} det(\lambda I - A_{ESO}(S_n)) &= \lambda det(\lambda I_{n-1} - \frac{n^2}{\lambda}(n^2-2n+2)J_{n-1}) \\ &= \lambda^{2-n} det(\lambda^2 I_{n-1} - n^2(n^2-2n+2)J_{n-1}). \end{aligned}$$

On the other hand, the eigenvalues of J_{n-1} are $n-1$ (once) and 0 ($n-2$ times), the eigenvalues of $n^2(n^2-2n+2)J_{n-1}$ are $(n-1)(n^2)(n^2-2n+2)$ (once) and 0 ($n-2$ times). Therefore

$$\phi_{ESO}(S_n,\lambda)) = \lambda^{n-2}\left(\lambda^2 - (n-1)(n^4-2n^3+2n^2)\right).$$

(ii) It follows from Part (i).

□

We close this section by computing the elliptic Sombor characteristic polynomial of complete bipartite graphs and their Sombor energy.

Theorem 2.5. *For natural number $m,n \neq 1$,*

(i) The elliptic Sombor characteristic polynomial of complete bipartite graph $K_{m,n}$ is

$$\phi_{ESO}(K_{m,n},\lambda) = \lambda^{m+n-2}(\lambda^2 - mn(m^2+n^2)).$$

(ii) The elliptic Sombor energy of $K_{m,n}$ is $2(m+n)\sqrt{mn(m^2+n^2)}$.

Proof. (i) It is easy to see that the elliptic Sombor matrix of $K_{m,n}$ is

$$(m+n)\sqrt{m^2+n^2}\begin{pmatrix} 0_{m\times m} & J_{m\times n} \\ J_{n\times m} & 0_{n\times n} \end{pmatrix}.$$

Using Lemma 2.3 we have

$$det(\lambda I - A_{ESO}(K_{m,n})) = det\begin{pmatrix} \lambda I_m & -(m+n)\sqrt{m^2+n^2}J_{m\times n} \\ -(m+n)\sqrt{m^2+n^2}J_{n\times m} & \lambda I_n \end{pmatrix}.$$

So $det(\lambda I - A_{ESO}(K_{m,n})) =$

$$det(\lambda I_m)det(\lambda I_n - (m+n)\sqrt{m^2+n^2}J_{n\times m}\frac{1}{\lambda}I_m(m+n)\sqrt{m^2+n^2}J_{m\times n}).$$

We know that $J_{n\times m}J_{m\times n} = mJ_n$. Therefore

$$\begin{aligned} det(\lambda I - A_{SO}(K_{m,n})) &= \lambda^m det(\lambda I_n - \frac{1}{\lambda}m(m^2+n^2)(m+n)^2 J_n) \\ &= \lambda^{m-n} det(\lambda^2 I_n - m(m^2+n^2)(m+n)^2 J_n). \end{aligned}$$

The eigenvalues of J_n are n (once) and 0 ($n-1$ times). So the eigenvalues of $m(m^2 + n^2)(m+n)^2 J_n$ are $mn(m^2+n^2)(m+n)^2$ (once) and 0 ($n-1$ times). Hence

$$\phi_{SO}(K_{m,n}, \lambda) = \lambda^{m+n-2}(\lambda^2 - mn(m^2+n^2)(m+n)^2).$$

(ii) It follows from Part (i).

□

3 Elliptic Sombor energy of k-regular graphs

In this section we consider 2-regular and 3-regular graphs. As a beginning of this section, we have the following easy lemma:

Lemma 3.1. *Let $G = G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_n$. Then*

(i) $\phi_{ESO}(G) = \prod_{i=1}^{n} \phi_{ESO}(G_i)$.

(ii) $E_{ESO}(G) = \sum_{i=1}^{n} E_{ESO}(G_i)$.

As an immediate result of Lemma 3.1, we have the following results:

Proposition 3.2. *(i) If $e = v_r v_{r+1} \in E(P_n)$, then $E_{ESO}(P_n - e) = E_{ESO}(P_r) + E_{ESO}(P_s)$, where $r + s = n$.*

(ii) If $e \in E(C_n)$, ($n \geq 3$), then $E_{ESO}(C_n - e) = E_{ESO}(P_n)$.

(iii) Let S_n be the star on n vertices and $e \in E(S_n)$. Then for any $n \geq 3$,

$$E_{ESO}(S_n - e) = E_{ESO}(S_{n-1}).$$

Now consider the 2-regulars. Every 2-regular graph is a disjoint union of cycles. By Theorem 2.2, we can find all the eigenvalues of elliptic Sombor matrix of cycle graphs. Therefore by Lemma 3.1, we can find elliptic Sombor characteristic polynomial and elliptic Sombor energy of 2-regular graphs. Before we continue, we need the following easy result that is the direct conclusion of the definition of elliptic Sombor energy:

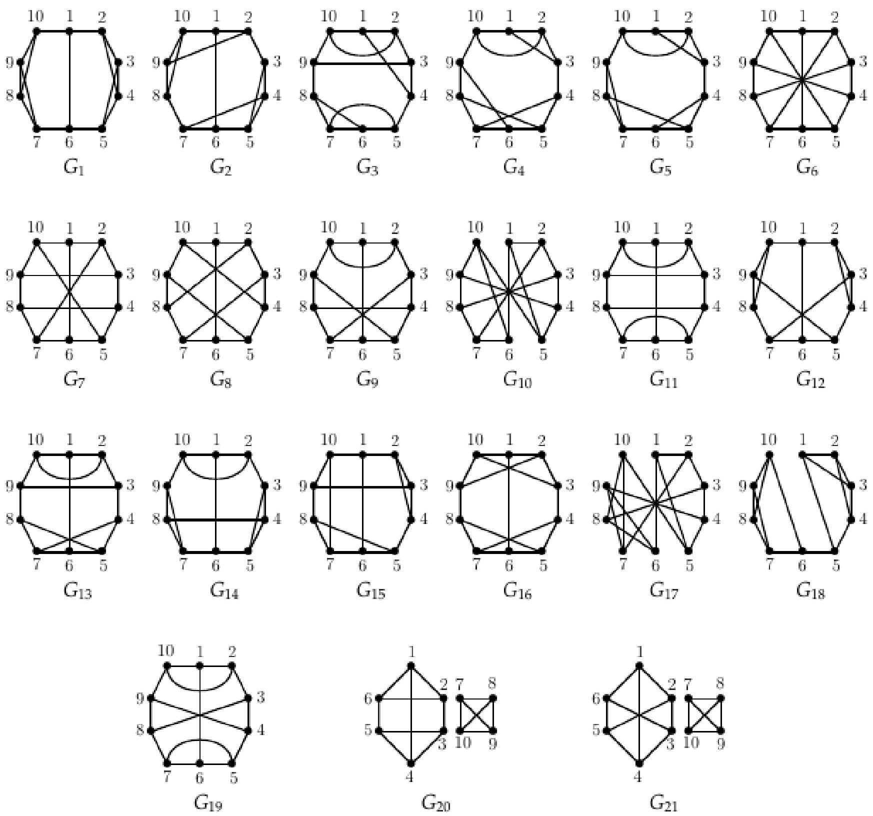

Figure 1. Cubic graphs of order 10.

Proposition 3.3. *If G is a k-regular graph of order n, then*

$$E_{ESO}(G) = (2k)^n E_{SO}(G).$$

In [16], we showed that the Sombor energy of K_n is $E_{SO}(K_n) = 2(n-1)^2\sqrt{2}$. So by Proposition 3.3, we have the following result:

Theorem 3.4. *For $n \geq 2$, The elliptic Sombor energy of K_n is*

$$E_{ESO}(K_n) = 2^{n+1}(n-1)^{n+2}\sqrt{2}.$$

Now, we consider to the elliptic characteristic polynomial of 3-regular graphs of order 10. Also we compute the elliptic Sombor energy of this class of graphs. There are exactly 21 cubic graphs of order 10 given in Figure 1 (see [15]). We have the Sombor energy of these graphs in table 1 as we computed them in [16]:

G_i	$E_{SO}(G_i)$	G_i	$E_{SO}(G_i)$	G_i	$E_{SO}(G_i)$
G_1	64.161	G_8	64.161	G_{15}	62.767
G_2	63.043	G_9	64.981	G_{16}	59.396
G_3	62.880	G_{10}	61.399	G_{17}	67.882
G_4	57.336	G_{11}	62.375	G_{18}	57.517
G_5	60.638	G_{12}	67.882	G_{19}	66.096
G_6	63.403	G_{13}	61.000	G_{20}	59.396
G_7	63.969	G_{14}	65.835	G_{21}	50.911

Table 1. Sombor energy of cubic graphs of order 10.

Now, by Using Table 1 and Proposition 3.3, we have the elliptic Sombor energy of cubic graphs of order 10 up to three decimal places, as we see in Table 2:

G_i	$E_{ESO}(G_i)$	G_i	$E_{ESO}(G_i)$	G_i	$E_{ESO}(G_i)$
G_1	13858.776	G_8	13858.776	G_{15}	13557.672
G_2	13617.288	G_9	14035.896	G_{16}	12829.536
G_3	13582.080	G_{10}	13262.184	G_{17}	14662.512
G_4	12384.576	G_{11}	13473.000	G_{18}	12423.672
G_5	13097.808	G_{12}	14662.512	G_{19}	12980.736
G_6	13695.048	G_{13}	13176.000	G_{20}	12829.536
G_7	13758.336	G_{14}	14220.360	G_{21}	10996.776

Table 2. Elliptic Sombor energy of cubic graphs of order 10.

Proposition 3.5. *Six cubic graphs of order* 10 *are not* $\mathcal{E}_{\mathcal{ESO}}$*-unique.*

Proof. By Table 2, we see that $[G_1] = \{G_1, G_8\}$, $[G_{12}] = \{G_{12}, G_{17}\}$ and $[G_{16}] = \{G_{16}, G_{20}\}$. Therefore, we have fifteen cubic graphs of order 10 which are $\mathcal{E}_{\mathcal{SO}}$-unique. □

As an immediate result of Proposition 3.5, we have:

Corollary 3.6. *In general, two k-regular graphs of the same order may have different elliptic Sombor energy.*

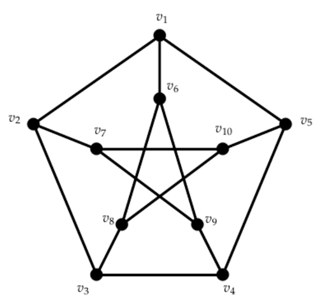

Figure 2. Petersen graph

Theorem 3.7. *Let $\mathcal{G}$ be the family of* 3*-regular graphs of order* 10*. For the Petersen graph P (Figure 2 or G_{17} in Figure 1), we have the following properties:*

(i) The Petersen graph P is not $\mathcal{E}_{\mathcal{ESO}}$-unique in $\mathcal{G}$.

(ii) The Petersen graph P has the maximum elliptic Sombor energy in $\mathcal{G}$.

Proof. (i) The Sombor matrix of P is

$$A_{ESO}(P)=\begin{pmatrix} 0 & 18\sqrt{2} & 0 & 0 & 18\sqrt{2} & 18\sqrt{2} & 0 & 0 & 0 & 0\\ 18\sqrt{2} & 0 & 18\sqrt{2} & 0 & 0 & 0 & 18\sqrt{2} & 0 & 0 & 0\\ 0 & 18\sqrt{2} & 0 & 18\sqrt{2} & 0 & 0 & 0 & 18\sqrt{2} & 0 & 0\\ 0 & 0 & 18\sqrt{2} & 0 & 18\sqrt{2} & 0 & 0 & 0 & 18\sqrt{2} & 0\\ 18\sqrt{2} & 0 & 0 & 18\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 18\sqrt{2}\\ 18\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 18\sqrt{2} & 18\sqrt{2} & 0\\ 0 & 18\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 18\sqrt{2} & 18\sqrt{2}\\ 0 & 0 & 18\sqrt{2} & 0 & 0 & 18\sqrt{2} & 0 & 0 & 0 & 18\sqrt{2}\\ 0 & 0 & 0 & 18\sqrt{2} & 0 & 18\sqrt{2} & 18\sqrt{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 18\sqrt{2} & 0 & 18\sqrt{2} & 18\sqrt{2} & 0 & 0 \end{pmatrix}.$$

So

$$\phi_{SO}(P,\lambda)=det(\lambda I-A_{SO}(P))=(\lambda-9\sqrt{2})(\lambda+6\sqrt{2})^4(\lambda-3\sqrt{2})^5.$$

Therefore we have:

$$\lambda_1=1994\sqrt{2}\ ,\ \lambda_2=\lambda_3=\lambda_4=\lambda_5=-1296\sqrt{2}\ ,\ \lambda_6=\lambda_7=\lambda_8=\lambda_9=\lambda_{10}=648\sqrt{2},$$

and so we have $E_{SO}(P)=10368\sqrt{2}$. By Table 2, we have $P\in\{G_{12},G_{17}\}$. Therefore P is not $\mathcal{E}_{ESO}$-unique in $\mathcal{G}$.

(ii) It follows from Part (i) and Table 2.

□

In [16], we have shown that if two connected k-regular graphs have the same Sombor energy, then their adjacency matrices may have or have not the same permanent. Now by Proposition 3.3, we have the following result:

Proposition 3.8. *If two connected k-regular graphs have the same elliptic Sombor energy, then their adjacency matrices may have or have not the same permanent.*

Also in [16], we have shown that if two graphs have the same permanent, then we can not conclude that they have same Sombor energy. Consequently, if we have two graphs with the same permanent, then we can not conclude that they have same elliptic Sombor energy.

We think that the elliptic Sombor energy of no graph is integer. We end this section with the following conjecture:

Conjecture 3.9. *There is no graph with integer-valued elliptic Sombor energy.*

4 Conclusions

In this paper we introduced the elliptic Sombor matrix and the elliptic Sombor energy of a graph G. We computed the elliptic Sombor characteristic polynomial and the elliptic Sombor energy for some graph classes. Also, we studied the elliptic Sombor energy of cubic graphs of order 10.

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