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Research Paper

Binary codes from uniform subset graph G(n, 4, i)

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Abstract. Take Ω to be a set with $n \geq 8$ elements, and set $V = P_4(\Omega)$ to be the collection of all 4-element subsets of Ω. For $i = 0,1,2,3$, let $G(n,4,i)$ be the graph with vertex set V, where two vertices are adjacent precisely when their intersection has cardinality i. This graph is referred to as the uniform subset graph. Let $A_i(n)$ represent the adjacency matrix associated with $G(n,4,i)$. In this work, we investigate the binary codes $C_i(n)$ arising from the row space of $A_i(n)$, determine their parameters, and establish the relationships between $C_i(n)$ and $C_j(n)^{\perp}$ for $i, j \in \{0,1,2,3\}$.

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1 Introduction

Let F_q^n be the n-dimensional vector space over the finite field $GF(q)$. A linear code C is defined as a subspace of F_q^n, whose elements are referred to as codewords. For a codeword, its weight is the number of nonzero entries in its coordinate representation. The smallest weight attained by a nonzero codeword is denoted by ω. A linear code C is represented as $[n,k,d]_q$, where $k = \dim(C)$ and d is the minimum distance. It is well known that for linear codes, $\omega = d$. A generator matrix for C is a $k \times n$ matrix whose rows constitute a basis of C. The dual code is $C^{\perp} = \{v \in F_q^n \mid v \cdot u = 0 \text{ for all } u \in C\}$. The support of a codeword c consists of the positions where c has nonzero entries. The all-one vector is denoted by j. For fundamental concepts and notation, we follow [5].

Let Ω be an n-set (whose elements are called points). The collection of all k-subsets of Ω

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is denoted by $P_k(\Omega)$. A pair $D = (\Omega, \beta)$, where $\beta \subseteq P_k(\Omega)$ has size b (the blocks), is called a t-(n,k,λ) design if every t-subset of Ω occurs in exactly λ blocks [1]. The incidence matrix of D is a $b \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ when the i-th block contains the j-th point, and 0 otherwise. Note that A depends on the ordering of blocks and points. The code $C_F(D)$ is the F-span of the rows of the incidence matrix over a finite field F. Equivalently, it is the row space of the incidence matrix. For $S \subseteq \Omega$, let v^S denote its incidence vector, with v^P used when $S = \{P\}$. Thus $C_F(D) = \langle v^B \mid B \in \beta \rangle$, a subspace of F_p. When $F = F_p$, we write $C_p(D)$ for $C_F(D)$.

Let $\Omega = \{v_1, \ldots, v_n\}$. A simple graph G of order n is a pair $G = (\Omega, E)$ with $E \subseteq P_2(\Omega)$. The graph has vertex set Ω and edge set E. Two distinct vertices u and v are connected by an edge precisely when $\{u, v\} \in E$. For each vertex v, its degree is given by the number of vertices adjacent to it. A graph is called regular when every vertex has the same degree. If the vertices of G are labeled as $v_1, \ldots, v_n$, we write its adjacency matrix as $A(G) = (a_{ij})$, where a_{ij} specifies the number of edges joining v_i and v_j.

Key, Moori, and Rodrigues investigated a family of regular graphs and examined binary codes derived from their adjacency matrices [6,7]. They subsequently studied triple codes associated with these graphs [8]. Let Ω have size n with $n \geq 8$, and let $V = \Omega^{(4)}$ be the set of all 4-subsets of Ω. For $i = 0,1,2,3$, the graph $G(n,4,i)$ has vertex set V, with two vertices adjacent exactly when their intersection has size i. This graph is known as the uniform subset graph [2,4]. We write $A_i(n)$ for its adjacency matrix. In this paper, we analyze the codes $C_i(n)$ arising from the row space of $A_i(n)$, determine their parameters, and establish the relationships between $C_i(n)$ and $C_j(n)^{\perp}$ for $i, j = 0,1,2,3$. See also [3,9].

2 Results

We begin by introducing some definitions and notation. For each $i \in \{0,1,2,3\}$, let $D_i(n)$ be the 1-design defined on the point set $\mathcal{P}$ from the matrix $A_i(n)$. To each point $P = \{a,b,c,d\} \in \mathcal{P}$, we assign a block

$$\overline{\{a,b,c,d\}}_i = \overline{P}_i = \{\{w,x,y,z\} \in \mathcal{P} \mid |\{w,x,y,z\} \cap \{a,b,c,d\}| = i\}.$$

The binary incidence vector of the block $\overline{P}_i = \overline{\{a,b,c,d\}}_i$ for $i = 0,1,2,3$ is given by

$$\begin{aligned}
\nu^{\overline{P}_0} &= \textstyle\sum_{w,x,y,z \in \Omega^*} \nu^{\{w,x,y,z\}}, \\
\nu^{\overline{P}_1} &= \textstyle\sum_{\substack{w \in \{a,b,c,d\} \\ x,y,z \in \Omega^*}} \nu^{\{w,x,y,z\}}, \\
\nu^{\overline{P}_2} &= \textstyle\sum_{\substack{w,x \in \{a,b,c,d\} \\ y,z \in \Omega^*}} \nu^{\{w,x,y,z\}}, \\
\nu^{\overline{P}_3} &= \textstyle\sum_{\substack{w,x,y \in \{a,b,c,d\} \\ z \in \Omega^*}} \nu^{\{w,x,y,z\}},
\end{aligned}$$

where $\Omega^* = \Omega \setminus \{a,b,c,d\}$. Moreover,

$$j = \nu^{\{a,b,c,d\}} + \nu^{\overline{\{a,b,c,d\}}_0} + \nu^{\overline{\{a,b,c,d\}}_1} + \nu^{\overline{\{a,b,c,d\}}_2} + \nu^{\overline{\{a,b,c,d\}}_3}. \tag{1}$$

Definition 2.1. *For a given point $P = \{a,b,c,d\} \in \mathcal{P}$ and for each $D_i(n)$, define the vector*

$$\omega_P = \sum_{P \in b_i} \nu^{b_i}, \tag{2}$$

where the b_i's are the blocks of $D_i(n)$. For an arbitrary point $Q \in \mathcal{P}$, the coordinate $\omega_P(Q)$ of ω_P at Q is given as follows:

- $i = 0$:
  1. If $P = Q$, then $\omega_P(Q) = \binom{n-4}{4}$.
  2. If $|P \cap Q| = 3$, then $\omega_P(Q) = \binom{n-5}{4}$, with $4\binom{n-4}{4}$ such Q's.
  3. If $|P \cap Q| = 2$, then $\omega_P(Q) = \binom{n-6}{4}$, with $6\binom{n-4}{2}$ such Q's.
  4. If $|P \cap Q| = 1$, then $\omega_P(Q) = \binom{n-7}{4}$, with $4\binom{n-4}{3}$ such Q's.
  5. If $|P \cap Q| = 0$, then $\omega_P(Q) = \binom{n-8}{4}$, with $\binom{n-4}{4}$ such Q's.
- $i = 1$:
  1. If $P = Q$, then $\omega_P(Q) = 4\binom{n-4}{3}$.
  2. If $|P \cap Q| = 3$, then $\omega_P(Q) = 3\binom{n-5}{3} + \binom{n-5}{2}$.
  3. If $|P \cap Q| = 2$, then $\omega_P(Q) = 2\binom{n-6}{3} + 4\binom{n-6}{2}$.
  4. If $|P \cap Q| = 1$, then $\omega_P(Q) = \binom{n-7}{3} + 9\binom{n-7}{2}$.
  5. If $|P \cap Q| = 0$, then $\omega_P(Q) = 16\binom{n-8}{2}$.
- $i = 2$:
  1. If $P = Q$, then $\omega_P(Q) = 6\binom{n-4}{2}$.
  2. If $|P \cap Q| = 3$, then $\omega_P(Q) = 3\left(\binom{n-5}{2} + (n-5)\right)$.
  3. If $|P \cap Q| = 2$, then $\omega_P(Q) = \binom{n-6}{2} + 8(n-6) + 1$.
  4. If $|P \cap Q| = 1$, then $\omega_P(Q) = 9(n-7) + 9$.
  5. If $|P \cap Q| = 0$, then $\omega_P(Q) = 36$.
- $i = 3$:
  1. If $P = Q$, then $\omega_P(Q) = 4(n-4)$.
  2. If $|P \cap Q| = 3$, then $\omega_P(Q) = n-2$.
  3. If $|P \cap Q| = 2$, then $\omega_P(Q) = 4$.
  4. If $|P \cap Q| = 1$, then $\omega_P(Q) = 0$.

5. *If* $|P \cap Q| = 0$, *then* $\omega_P(Q) = 0$.

The preceding definition and observations lead to the following two lemmas.

Lemma 2.2. *Let* $P = \{a,b,c,d\} \in \mathcal{P}$. *Then:*

1. *If* $n \equiv 0 \pmod 4$*:*
   (a) *In* $C_0(n)$*: for* $n \equiv 0 \pmod 8$, $\omega_P = \nu^P$*; for* $n \equiv 4 \pmod 8$, $\omega_P = j - \nu^P$.
   (b) *In* $C_1(n)$, $C_2(n)$, *and* $C_3(n)$, $\omega_P = \vec{0}$.

2. *If* $n \equiv 1 \pmod 4$*:*
   (a) *In* $C_0(n)$*: for* $n \equiv 1 \pmod 8$, $\omega_P = \nu^P + \nu^{\overline{\{a,b,c,d\}}_3}$*; for* $n \equiv 5 \pmod 8$, $\omega_P = \nu^{\overline{\{a,b,c,d\}}_2} + \nu^{\overline{\{a,b,c,d\}}_1} + \nu^{\overline{\{a,b,c,d\}}_0}$.
   (b) *In* $C_1(n)$, $\omega_P = \nu^{\overline{\{a,b,c,d\}}_1}$.
   (c) *In* $C_2(n)$, $\omega_P = \nu^{\overline{\{a,b,c,d\}}_2}$.
   (d) *In* $C_3(n)$, $\omega_P = \nu^{\overline{\{a,b,c,d\}}_3}$.

3. *If* $n \equiv 2 \pmod 4$*:*
   (a) *In* $C_0(n)$*: for* $n \equiv 2 \pmod 8$, $\omega_P = \nu^P + \nu^{\overline{\{a,b,c,d\}}_3} + \nu^{\overline{\{a,b,c,d\}}_2}$*; for* $n \equiv 6 \pmod 8$, $\omega_P = \nu^{\overline{\{a,b,c,d\}}_1} + \nu^{\overline{\{a,b,c,d\}}_0}$.
   (b) *In* $C_1(n)$, $\omega_P = \vec{0}$.
   (c) *In* $C_2(n)$, $\omega_P = \nu^{\overline{\{a,b,c,d\}}_3} + \nu^{\overline{\{a,b,c,d\}}_2} + \nu^{\overline{\{a,b,c,d\}}_1}$.
   (d) *In* $C_3(n)$, $\omega_P = \vec{0}$.

4. *If* $n \equiv 3 \pmod 4$*:*
   (a) *In* $C_0(n)$*: for* $n \equiv 3 \pmod 8$, $\omega_P = \nu^P + \nu^{\overline{\{a,b,c,d\}}_3} + \nu^{\overline{\{a,b,c,d\}}_2} + \nu^{\overline{\{a,b,c,d\}}_1}$*; for* $n \equiv 7 \pmod 8$, $\omega_P = \nu^{\overline{\{a,b,c,d\}}_0}$.
   (b) *In* $C_1(n)$, $\omega_P = \nu^{\overline{\{a,b,c,d\}}_3}$.
   (c) *In* $C_2(n)$, $\omega_P = \nu^{\overline{\{a,b,c,d\}}_3} + \nu^{\overline{\{a,b,c,d\}}_2} + \nu^{\overline{\{a,b,c,d\}}_1}$.
   (d) *In* $C_3(n)$, $\omega_P = \nu^{\overline{\{a,b,c,d\}}_3}$.

Proof. For part 1(a), when $i = 0$ and $n \equiv 0 \pmod 8$, we have $\omega_P(Q) \equiv 1 \pmod 2$ when $P = Q$, and $\omega_P(Q) \equiv 0 \pmod 2$ when $|P \cap Q| = 0,1,2,3$. Hence $\omega_P = \nu^P$. When $n \equiv 4 \pmod 8$, $\omega_P(Q) \equiv 0 \pmod 2$ when $P = Q$, and $\omega_P(Q) \equiv 1 \pmod 2$ otherwise, giving $\omega_P = j - \nu^P$. For part 1(b), in $C_1(n)$, $C_2(n)$, and $C_3(n)$, $\omega_P(Q) \equiv 0 \pmod 2$ for all cases, so $\omega_P = \vec{0}$. The remaining cases follow similarly. □

Lemma 2.3. *For every* $n \geq 8$*:*

1. *If* $n \equiv 0 \pmod 4$*:*

   - *When* $n \equiv 0 \pmod 8$*,* $C_0(n) = F_2^{\mathcal{P}}$*.*
   - *When* $n \equiv 4 \pmod 8$*,* $\dim C_0(n) = \binom{n}{4} - 1$ *and* $C_0(n) = \langle j \rangle^{\perp}$*.*

2. *For* $n \equiv 1 \pmod 4$*,* $C_1(n) \subseteq C_2(n)$*. When* $n \equiv 1 \pmod 8$*,* $C_0(n) + C_3(n) = F_2^{\mathcal{P}}$*. When* $n \equiv 5 \pmod 8$*,* $C_0(n) + C_3(n) = \langle j \rangle^{\perp}$*.*

3. *For* $n \equiv 2 \pmod 4$*,* $C_1(n) \subseteq C_0(n)$*. When* $n \equiv 2 \pmod 8$*,* $C_0(n) + C_2(n) = F_2^{\mathcal{P}}$*. When* $n \equiv 6 \pmod 8$*,* $C_0(n) + C_2(n) = \langle j \rangle^{\perp}$*.*

4. *For* $n \equiv 3 \pmod 4$*,* $C_3(n) \subseteq C_1(n)$*. When* $n \equiv 3 \pmod 8$*,* $C_0(n) + C_2(n) = F_2^{\mathcal{P}}$*. When* $n \equiv 7 \pmod 8$*,* $C_0(n) + C_2(n) = \langle j \rangle^{\perp}$*.*

Proof. For $n \equiv 0 \pmod 8$, we have $\omega_P = v^P$ in $C_0(n)$, so $C_0(n)$ contains all weight-1 vectors and hence $C_0(n) = F_2^{\mathcal{P}}$. For $n \equiv 4 \pmod 8$, $\omega_P = j - v^P$, so $\dim C_0(n) = \binom{n}{4} - 1$. Since $\binom{n-4}{4}$ is even, all rows of $A_0(n)$ have even weight, and since $\binom{n}{4}$ is odd, j has odd weight, so $j \notin C_0(n)$. Also j is orthogonal to all rows of $A_0(n)$, hence $j \in C_0(n)^{\perp}$ and $C_0(n) = \langle j \rangle^{\perp}$. The other cases are analogous. □

Definition 2.4. *For distinct points* $a, b, c \in \Omega$*, define*

$$\omega(a) := \sum_{x,y,z \in \Omega \setminus \{a\}} v^{\{a,x,y,z\}},$$

$$\omega(a,b) := \sum_{x,y \in \Omega \setminus \{a,b\}} v^{\{a,b,x,y\}},$$

$$\omega(a,b,c) := \sum_{x \in \Omega \setminus \{a,b,c\}} v^{\{a,b,c,x\}}.$$

The weights of $\omega(a)$*,* $\omega(a,b)$*, and* $\omega(a,b,c)$ *are respectively* $\binom{n-1}{3}$*,* $\binom{n-2}{2}$*, and* $(n-3)$*. Define*

$$W_1 := \langle \omega(a) \rangle, \quad W_2 := \langle \omega(a,b) \rangle, \quad W_3 := \langle \omega(a,b,c) \rangle.$$

Lemma 2.5. *We have:*

1. $W_1 \subseteq W_2$*,* $W_1 \subseteq W_3$*, and* $\dim(W_1) = n - 1$*.*

2. $\dim(W_2) = \binom{n}{2} - 1$*.*

3. $\dim(W_3) = \binom{n-1}{3}$*.*

4. $\dim\langle W_2 + W_3 \rangle = \binom{n}{3} - 1$ *and* $W_2 \cap W_3 = W_1$*.*

Proof. (1) Since $\sum_{x\in\Omega\setminus\{a\}}\omega(a,x)=\omega(a)$, we have $W_1\subseteq W_2$. Also $\sum_{x,y\in\Omega\setminus\{a\}}\omega(a,x,y)=\omega(a)$, so $W_1\subseteq W_3$. Since $\sum_{a\in\Omega}\omega(a)=4j=0$, we have $\dim(W_1)\leq n-1$. Let $\omega(n)=\sum_{a\neq n}\omega(a)$ and suppose $\sum_{i=1}^{n}\alpha_i\omega(i)=0$ with $\alpha_n=0$. Then for all $i,j,k,l\in\Omega$, the coefficient of $\nu^{\{i,j,k,l\}}$ is $\alpha_i+\alpha_j+\alpha_k+\alpha_l=0$. For $\nu^{\{i,j,k,n\}}$, the coefficient is $\alpha_i+\alpha_j+\alpha_k=0$, implying $\alpha_l=0$. Thus all coefficients vanish and $\dim(W_1)=n-1$.

(2) There are $\binom{n}{2}$ vectors of the form $\omega(a,b)$. Since $\sum_{a,b\in\Omega}\omega(a,b)=6j=0$, let $\omega(n-1,n)=\sum_{\{a,b\}\neq\{n-1,n\}}\omega(a,b)$ and suppose $\sum_{i,j}\alpha_{i,j}\omega(i,j)=0$ with $\alpha_{n-1,n}=0$. The coefficient of $\nu^{\{i,j,k,l\}}$ is $\alpha_{i,j}+\alpha_{i,k}+\alpha_{i,l}+\alpha_{j,k}+\alpha_{j,l}+\alpha_{k,l}=0$. By considering sums of coefficients for $\nu^{\{a,b,c,d\}}$, $\nu^{\{a,b,c,e\}}$, and $\nu^{\{b,c,d,e\}}$, we get $\alpha_{a,d}+\alpha_{a,e}+\alpha_{b,c}+\alpha_{d,e}=0$. Setting $b=n-1$ and $c=n$ gives $\alpha_{a,d}+\alpha_{a,e}+\alpha_{d,e}=0$, and for all $b,c\in\Omega\setminus\{a,d,e\}$, $\alpha_{b,c}=0$. Repeating this argument shows all remaining coefficients are zero. Hence these $\binom{n}{2}-1$ vectors are linearly independent and $\dim(W_2)=\binom{n}{2}-1$.

(3) There are $\binom{n}{3}$ vectors of the form $\omega(a,b,c)$. Form a matrix of size $\binom{n-1}{3}\times\binom{n}{4}$ whose columns are indexed by 4-subsets and rows by $\omega(a,b,c)$. Order the columns as $\{n-3,n-2,n-1,n\},\ldots,\{1,3,4,n\},\{1,2,n-1,n\},\ldots,\{1,2,4,n\},\{1,2,3,n\}$ and the rows as $\omega(n-3,n-2,n-1),\ldots,\omega(1,3,4),\omega(1,2,n-1),\ldots,\omega(1,2,4),\omega(1,2,3)$. The resulting matrix is the identity matrix. Therefore $\dim(W_3)\geq\binom{n-1}{3}$, and since any other vector in W_3 is of the form $\omega(a,b,n)=\sum_{x\neq a,b,n}\omega(a,b,x)$, we get $\dim(W_3)=\binom{n-1}{3}$.

(4) Each $\omega(a,n)$ can be written as $\sum_{x,y\neq a,n}(\omega(a,x,y)+\omega(a,x))=u$. Points forming $P=\{a,b,c,n\}\in\operatorname{supp}(\omega(a,n))$ appear in the support of $\omega(a,c)$, $\omega(a,b)$, and $\omega(a,b,c)$, hence $P\in\operatorname{supp}(u)$. Points $Q=\{a,b,c,d\}\notin\operatorname{supp}(\omega(a,n))$ appear in $\omega(a,b),\omega(a,c),\omega(b,c),\omega(a,b,c)$, $\omega(a,b,d)$, $\omega(a,c,d)$, so $Q\notin\operatorname{supp}(u)$. Other points are not in the support of $\omega(a,n)$ or any constituent vectors of u. Thus $\operatorname{supp}(\omega(a,n))=\operatorname{supp}(u)$, so $u=\omega(a,n)$. We also have
$\sum_{a,b,c\in\Omega\setminus\{n\}}(\omega(a,b,c)+\omega(a,b))=10\sum_{a,b,c,d\in\Omega\setminus\{n\}}\nu^{\{a,b,c,d\}}+4\sum_{a,b,c\in\Omega\setminus\{n\}}\nu^{\{a,b,c,n\}}=0.$

Thus $\omega(n-2,n-1)$ is dependent on other vectors and $\dim\langle W_2+W_3\rangle\leq\binom{n}{2}-1-n+\binom{n-1}{3}=\binom{n}{3}-1$. Consider a linear combination of these $\binom{n}{3}-1$ vectors with $\alpha_{n-2,n-1}=0$: $\sum_{a,b,c\in\Omega\setminus\{n\}}(\alpha_{a,b,c}\omega(a,b,c)+\alpha_{a,b}\omega(a,b))=0$. The coefficient of each $\nu^{\{i,j,k,l\}}$ is $\alpha_{i,j,k}+\alpha_{i,j,l}+\alpha_{i,k,l}+\alpha_{j,k,l}+\alpha_{i,j}+\alpha_{i,k}+\alpha_{i,l}+\alpha_{j,k}+\alpha_{j,l}+\alpha_{k,l}=0$. Summing the coefficients of $\nu^{\{a,c,d,e\}}$, $\nu^{\{a,b,d,e\}}$, and $\nu^{\{a,b,c,e\}}$ gives $\alpha_{a,c,d}+\alpha_{b,c,d}+\alpha_{a,c,e}+\alpha_{b,c,e}+\alpha_{a,d,e}+\alpha_{b,d,e}+\alpha_{c,d}+\alpha_{c,e}+\alpha_{a,b}+\alpha_{d,e}=0$. The coefficient of $\nu^{\{a,b,c,n\}}$ is $\alpha_{a,b,c}+\alpha_{a,c}+\alpha_{a,b}+\alpha_{b,c}=0$. Substituting yields $\alpha_{c,d}+\alpha_{a,b}+\alpha_{d,e}+\alpha_{c,e}=0$. Taking $a=n-2$, $b=n-1$ gives $\alpha_{c,d}+\alpha_{c,e}+\alpha_{d,e}=0$ and $\alpha_{a,b}=\alpha_{n-2,n-1}=0$. Thus for $a,b\in\Omega\setminus\{c,d,e\}$, $\alpha_{a,b}=0$. The other $\alpha_{i,j}$ are similarly zero, implying all $\alpha_{i,j,k}=0$. It follows that the $\binom{n}{3}-1$ vectors are linearly independent, and therefore $\dim\langle W_2+W_3\rangle=\binom{n}{3}-1$. Also since $\dim(W_1\cap W_2)=n-1=\dim(W_1)$ and $W_1\subseteq W_2\cap W_3$, we have $W_2\cap W_3=W_1$. □

Lemma 2.6. *For each $n\geq 8$ and distinct points $a,b,c\in\Omega$:*

(1) $\omega(a,b)\in C_0(n)^{\perp}$ iff $n\equiv 2,3 \pmod 4$; $\omega(a,b,c)\in C_0(n)^{\perp}$ iff $n\equiv 1,3 \pmod 4$.

(2) $\omega(a,b)\in C_1(n)^{\perp}$ iff $n\equiv 1,2 \pmod 4$; $\omega(a,b,c)\in C_1(n)^{\perp}$ iff $n\equiv 0,2 \pmod 4$.

(3) $\omega(a,b)\in C_2(n)^{\perp}$ iff $n\equiv 1 \pmod 4$; for all n, $\omega(a,b,c)\notin C_2(n)^{\perp}$.

(4) $\omega(a,b,c) \in C_3(n)^{\perp}$ *iff* $n \equiv 0,2 \pmod 4$*; for all* n, $\omega(a,b) \notin C_3(n)^{\perp}$.

Proof. We determine the conditions under which the inner product of $\omega(a,b)$ or $\omega(a,b,c)$ with a row of the incidence matrix vanishes modulo 2.

(1) For $C_0(n)$ and $\omega(a,b)$: If $|\{a,b\} \cap \{c,d,e,f\}| = 2$ or 1, there are no common 1 entries. If $|\{a,b\} \cap \{c,d,e,f\}| = 0$, then $\omega(a,b) \cdot v^{\overline{\{c,d,e,f\}}_0} = \binom{n-6}{2}$, which is zero when $n \equiv 2,3 \pmod 4$. For $\omega(a,b,c)$: If $|\{a,b,c\} \cap \{d,e,f,g\}| = 1,2,3$, there are no common 1 entries. If the intersection is empty, then $\omega(a,b,c) \cdot v^{\overline{\{d,e,f,g\}}_0} = n-7$, which is zero for $n \equiv 1,3 \pmod 4$.

(3) The vectors $v^{\overline{\{a,e,f,g\}}_2} \in C_2(n)$ and $\omega(a,b,c)$ have 1's only at positions $\{a,b,c,e\}$, $\{a,b,c,f\}$, and $\{a,b,c,g\}$. Thus their inner product is not zero modulo 2, so $\omega(a,b,c) \notin C_2(n)^{\perp}$. The other cases are proved similarly. $\square$

Note that if $n \equiv 4 \pmod 8$, then $C_0(n) = \langle j \rangle^{\perp}$. Therefore, $C_0(n) = [\binom{n}{4}, \binom{n}{4} - 1, 2]_2$.

Theorem 2.7. *If n is odd and $n \geq 8$, then $C_3(n) = [\binom{n}{4}, \binom{n-1}{3}, n-3]_2$. If n is even, then $C_3(n) = [\binom{n}{4}, \binom{n-2}{3}, 4(n-4)]_2$.*

Proof. Let n be odd. Each row of the incidence matrix of $D_3(n)$ can be written as

$$v^{\overline{\{a,b,c,d\}}_3} = \omega(a,b,c) + \omega(a,b,d) + \omega(a,c,d) + \omega(b,c,d). \tag{3}$$

Thus $C_3(n) \subseteq W_3(n)$. Since n is odd,

$$\sum_{x \neq a,b,c} v^{\overline{\{a,b,c,x\}}_3} = (n-4) \sum_{x \neq a,b,c} v^{\{a,b,c,x\}} + 2 \sum_{\substack{i,j \in \{a,b,c\} \\ x,y \neq \{a,b,c\}}} v^{\{i,j,x,y\}} = \sum_{x \in \Omega \setminus \{a,b,c\}} v^{\{a,b,c,x\}} = \omega(a,b,c).$$

Therefore $W_3(n) \subseteq C_3(n)$, so $W_3(n) = C_3(n)$. By Lemma 2.5, $\dim(C_3(n)) = \binom{n-1}{3}$. Each row of the incidence matrix of $W_3(n)$ has weight $n-3$. Suppose a linear combination of r vectors of the form $\{\omega(a,b,x) \mid x \in \Omega \setminus \{a,b,n\}\}$ has weight less than $n-3$. Then $r(n-3) - 2\binom{r}{2} < n-3$, implying $r < 1$ or $r > n-3$, both impossible since $r \leq n-3$. Thus $w(C_3(n)) = n-3$ and $C_3(n) = [\binom{n}{4}, \binom{n-1}{3}, n-3]_2$. For even n, a similar argument as in Lemma 2.5 gives $\dim(C_3(n)) = \binom{n-2}{3}$, and the minimum weight is given in [4]. $\square$

Theorem 2.8. *If $n \geq 8$, then $C_1(n) \subseteq C_3(n)$. If $n \equiv 0,3 \pmod 4$, then $C_1(n) = C_3(n)$.*

Proof. For every $\{a,b,c,d,e\}$,

$$v^{\overline{\{a,b,c,d\}}_1} = \sum_{x,y \neq a,b,c,d,e} \left(v^{\overline{\{a,e,x,y\}}_3} + v^{\overline{\{b,e,x,y\}}_3} + v^{\overline{\{c,e,x,y\}}_3} + v^{\overline{\{d,e,x,y\}}_3} \right).$$

Hence $C_1(n) \subseteq C_3(n)$. If $n \equiv 3 \pmod 4$, Lemma 2.3 gives $C_3(n) \subseteq C_1(n)$, so equality holds. If $n \equiv 0 \pmod 4$, for every $\{a,b,c,d,e\}$,

$$\sum_{x,y,z \in \Omega \setminus \{a,b,c,d,e\}} \left(v^{\overline{\{a,x,y,z\}}_1} + v^{\overline{\{b,x,y,z\}}_1} + v^{\overline{\{c,x,y,z\}}_1} + v^{\overline{\{d,x,y,z\}}_1} \right) = v^{\overline{\{a,b,c,d\}}_3}.$$

Thus $C_3(n) \subseteq C_1(n)$ and equality holds. $\square$

Theorem 2.9. *If $n \equiv 0 \pmod 4$, then $C_1(n) = [\binom{n}{4}, \binom{n-2}{3}, 4(n-4)]_2$. If $n \equiv 3 \pmod 4$, then $C_1(n) = [\binom{n}{4}, \binom{n-1}{3}, n-3]_2$.*

Proof. This follows directly from Theorems 2.7 and 2.8. □

Theorem 2.10. *If $n \equiv 1 \pmod 4$, then $C_1(n) = [\binom{n}{4}, \binom{n}{3} - \binom{n}{2}]_2$. If $n \equiv 2 \pmod 4$, then $C_1(n) = [\binom{n}{4}, \binom{n-1}{3} - \binom{n-1}{2}]_2$.*

Proof. First, we show that $C_1^{\perp}$ contains weight-5 codewords. Let $\Delta_5 = \{a,b,c,d,e\}$ be a 5-subset of Ω. Consider

$$u(\Delta_5) = v^{\{a,b,c,d\}} + v^{\{a,b,c,e\}} + v^{\{a,b,d,e\}} + v^{\{a,c,d,e\}} + v^{\{b,c,d,e\}}.$$

If $u(\Delta_5) \cdot v^{\overline{\{w,x,y,z\}}_1} \equiv k \pmod 2$, then:

- If $|\{w,x,y,z\} \cap \Delta_5| = 4, 3$, or 0, then $k = 0$.
- If $|\{w,x,y,z\} \cap \Delta_5| = 2$, say $\{w,x,y,z\} = \{a,b,y,z\}$, then $k = 2$.
- If $|\{w,x,y,z\} \cap \Delta_5| = 1$, say $\{w,x,y,z\} = \{a,x,y,z\}$, then $k = 4$.

Thus $u(\Delta_5) \in C_1^{\perp}(n)$. Now $\dim\langle u(\Delta_5)\rangle = \binom{n-1}{4}$. Consider the matrix of order $\binom{n-1}{4} \times \binom{n}{4}$ with columns indexed by points and rows indexed by $u(\Delta_5)$. The resulting matrix is the identity, so $\dim\langle u(\Delta_5)\rangle \geq \binom{n-1}{4}$. Other weight-5 codewords satisfy the same relation, hence $\dim\langle u(\Delta_5)\rangle = \binom{n-1}{4}$ and $\dim(C_1^{\perp}(n)) \geq \binom{n-1}{4}$.

If $n \equiv 1 \pmod 4$, Lemma 2.6 gives $\omega(a,b) \in C_1^{\perp}(n)$. The $n-1$ independent vectors $\omega(1,n), \omega(2,n), \ldots, \omega(n-1,n)$ have no 1's in their $\binom{n-1}{4}$ corresponding entries, so they are independent of the weight-5 codewords. Thus $\dim(C_1^{\perp}(n)) \geq \binom{n-1}{4} + (n-1)$, giving $\dim(C_1(n)) \leq \binom{n-1}{3} - (n-1)$. Also, for $\Omega' = \Omega \setminus \{a,b,c\}$,

$$\sum_{x \neq a,b,c} v^{\overline{\{a,b,c,x\}}_1} + \omega(a) + \omega(b) + \omega(c) = \omega(a,b,c).$$

Thus $C_1(n) + W_1 = W_3$, so $\dim(C_1(n)) \geq \binom{n-1}{3} - (n-1)$. Hence equality holds and $\dim(C_1(n)) = \binom{n-1}{3} - (n-1) = \binom{n}{3} - \binom{n}{2}$.

If $n \equiv 2 \pmod 4$, Lemma 2.6 gives $\omega(a,b)$ and $\omega(a,b,c) \in C_1^{\perp}(n)$. The $\binom{n-2}{2}$ vectors of the form $\omega(a,b,c)$ and $n-2$ vectors of the form $\omega(a,n)$ (where $a,b \in \Omega \setminus \{n-1,n\}$) are linearly independent by Lemma 2.5 and have no 1's in their $\binom{n-1}{4}$ corresponding entries. Thus $\dim(C_1^{\perp}(n)) \geq \binom{n-1}{4} + \binom{n-1}{2}$, giving $\dim(C_1(n)) \leq \binom{n-1}{3} - \binom{n-1}{2}$. For $a,b,c \in \Omega \setminus \{n-1,n\}$,

$$v^{\overline{\{a,b,c,n-1\}}_3} = \omega(a,b,c) + \omega(a,b,n-1) + \omega(a,c,n-1) + \omega(b,c,n-1).$$

Then $C_1(n) + \langle \omega(1) + \omega(n-1), \omega(2) + \omega(n-1), \ldots, \omega(n-2) + \omega(n-1)\rangle = C_3(n)$. Hence $\dim(C_1(n)) \geq \binom{n-2}{3} - (n-2) = \binom{n-1}{3} - \binom{n-1}{2}$, proving the claim. □

Theorem 2.11. *If $n \equiv 1 \pmod 8$, then $C_0(n) = [\binom{n}{4}, \binom{n-1}{4}, 5]_2$. If $n \equiv 5 \pmod 8$, then $C_0(n) = [\binom{n}{4}, \binom{n-1}{4} - 1]_2$.*

Proof. By Theorem 2.7, if $n \equiv 1 \pmod 4$, then $\omega(a,b,c) \in C_0^{\perp}(n)$ and $C_3(n) = W_3(n)$, so $C_3(n) \subseteq C_0^{\perp}(n)$. If $n \equiv 1 \pmod 8$, Lemma 2.3 gives $C_0(n) + C_3(n) = F_2^{\mathcal{P}}$ and $C_0^{\perp}(n) = C_3(n)$. Hence $\dim(C_0^{\perp}(n)) = \dim(C_3(n)) = \binom{n-1}{3}$, $\dim(C_0^{\perp}(n) \cap C_0(n)) = 0$, and $\dim(C_0(n)) = \binom{n-1}{4}$. The weight-5 codeword

$$v^{\overline{\{a,b,c,d,e\}}} = v^{\{a,b,c,d\}} + v^{\{a,b,c,e\}} + v^{\{a,b,d,e\}} + v^{\{a,c,d,e\}} + v^{\{b,c,d,e\}}$$

lies in $C_0(n)$. For $n = 9$, SageMath confirms $w(C_0(n)) = 5$; for $n > 9$, it is known [8] that $w(C_0(n)) \geq 5$. Thus $w(C_0(n)) = 5$. If $n \equiv 5 \pmod 8$, then the weight of $\omega(a,b,c)$ is $n-3$ (even), and the weight of j is $\binom{n}{4}$ (odd). Therefore $j \notin W_3(n) = C_3(n)$. We have $j \in C_3^{\perp}(n)$, $j \notin C_0(n)$, and $j \in C_0^{\perp}(n)$. By Lemma 2.3, $C_0(n) + C_3(n) + j = F_2^{\mathcal{P}}$. Since $C_3(n) + j \subseteq C_0^{\perp}(n)$, we get $C_0(n) + C_0^{\perp}(n) = F_2^{\mathcal{P}}$ and $\langle C_3(n) + j\rangle = C_0^{\perp}(n)$. Hence $\dim(C_0(n)) = \binom{n-1}{4} - 1$ and $\dim(C_0^{\perp}(n)) = \binom{n-1}{3} + 1$. □

Theorem 2.12. *If $n \equiv 3 \pmod 4$ and $n \geq 11$, then $C_2(n) = [\binom{n}{4}, \binom{n}{3} - 1]_2$. If $n \equiv 3 \pmod 8$, then $C_0(n) = [\binom{n}{4}, \binom{n}{4} - \binom{n}{3} + 1]_2$. If $n \equiv 7 \pmod 8$, then $C_0(n) = [\binom{n}{4}, \binom{n}{4} - \binom{n}{3}]_2$.*

Proof. For $n \geq 8$ and all $a,b,c,d \in \Omega$,

$$\begin{aligned} v^{\overline{\{a,b,c,d\}}_2} = {} & \omega(a,b) + \omega(a,c) + \omega(b,c) + \omega(a,d) + \omega(b,d) + \omega(c,d) \\ & + \omega(a,b,c) + \omega(a,b,d) + \omega(a,c,d) + \omega(b,c,d). \end{aligned} \tag{1}$$

Thus $C_2(n) \subseteq \langle W_2 + W_3\rangle$ and $\dim(C_2(n)) \leq \binom{n}{3} - 1$. For $n \equiv 3 \pmod 4$, we show $C_2(n) \subseteq C_0(n)^{\perp}$. If $v^{\overline{\{a,b,c,d\}}_2} \cdot v^{\overline{\{e,f,g,h\}}_0} \equiv k \pmod 2$:

- If $|\{a,b,c,d\} \cap \{e,f,g,h\}| = 3$ or 4, then $k = 0$.
- If the intersection size is 2, say $\{e,f,g,h\} = \{a,b,g,h\}$, then the common 1's occur at entries $\{c,d,x,y\}$ with $x,y \neq a,b,c,d,g,h$. The count is $\binom{n-6}{2}$, which is $0 \pmod 2$ when $n \equiv 3 \pmod 4$.
- If the intersection size is 1, $k = 3\binom{n-7}{2} \equiv 0 \pmod 2$.
- If the intersection is empty, $k = 6\binom{n-8}{2} \equiv 0 \pmod 2$.

Thus $C_2(n) \subseteq C_0(n)^{\perp}$. If $n \equiv 3 \pmod 8$, Lemma 2.3 gives $C_0(n) + C_2(n) = F_2^{\mathcal{P}}$, so $C_0(n) \cap C_0(n)^{\perp} = \{0\}$ and $C_2(n) = C_0(n)^{\perp}$. If $n \equiv 7 \pmod 8$, then $C_0(n) + C_2(n) + j = F_2^{\mathcal{P}}$. Since $j \in C_0(n)^{\perp}$, $j \notin C_0(n)$, and $j \notin C_2(n)$, we have $j + C_2(n) = C_0(n)^{\perp}$. Since $W_2, W_3 \subseteq C_0(n)^{\perp}$, Lemma 2.5 gives $\dim(C_0(n)^{\perp}) \geq \dim\langle W_2 + W_3\rangle = \binom{n}{3} - 1$. Therefore, for $n \equiv 3 \pmod 8$, $\dim(C_0(n)) = \binom{n}{4} - \binom{n}{3} + 1$ and $\dim(C_2(n)) = \binom{n}{3} - 1$. For $n \equiv 7 \pmod 8$, since W_2 has even-weight codewords and W_3 has odd-weight codewords, $j \notin \langle W_2 + W_3\rangle$. Hence $\dim(C_2(n)) = \binom{n}{3} - 1$, $W_2 + W_3 \subseteq C_2(n)$, and $j \in C_2(n)$. This gives $\dim(C_2(n)) = \binom{n}{3}$, $\dim(C_0(n)) = \binom{n}{4} - \binom{n}{3}$, and $\dim(C_0(n)^{\perp}) = \binom{n}{3}$. □

Theorem 2.13. *If $n \equiv 2 \pmod 8$, then $C_0(n) = [\binom{n}{4}, \binom{n}{4} - \binom{n}{2} + 1]_2$. If $n \equiv 6 \pmod 8$, then $C_0(n) = [\binom{n}{4}, \binom{n}{4} - \binom{n}{2}]_2$.*

Proof. By Lemma 2.6, $W_2 \subseteq C_0(n)^{\perp}$. Clearly,

$$\omega' = v^{\overline{\{a,b,c,d\}}_3} + v^{\overline{\{a,b,c,d\}}_2} = \omega(a,b) + \omega(a,c) + \omega(b,c) + \omega(a,d) + \omega(b,d) + \omega(c,d) \in W_2.$$

Let $P = \{a,b,c,d\}$ and $n \equiv 2 \pmod 8$. By Lemma 2.3, $\omega' + \omega_P = v^P \in W_2 + C_0(n)$, so $C_0(n) + W_2 = F_2^{\mathcal{P}}$. Thus $C_0(n)^{\perp} = W_2$ and $\dim(C_0(n) \cap C_0(n)^{\perp}) = 0$, giving $\dim(C_0(n)^{\perp}) = \binom{n}{2} - 1$ and $\dim(C_0(n)) = \binom{n}{4} - \binom{n}{2} + 1$. If $n \equiv 6 \pmod 8$, then $j \notin C_0(n)$ and $j \notin W_2$, but $j \in C_0(n)^{\perp}$. By Lemma 2.3, $\omega_P + (\omega' + j) = v^P \in C_0(n) + \langle W_2 + j \rangle$, so $C_0(n) + \langle W_2 + j \rangle = F_2^{\mathcal{P}}$. Therefore $C_0(n)^{\perp} = \langle W_2 + j \rangle$, $\dim(C_0(n)^{\perp}) = \binom{n}{2}$, and $\dim(C_0(n)) = \binom{n}{4} - \binom{n}{2}$. □

Theorem 2.14. *If $n \equiv 2 \pmod 4$, then $C_2(n) = [\binom{n}{4}, \binom{n}{3} - \binom{n-2}{2}]_2$.*

Proof. From equations (3) and (1),

$$v^{\overline{\{a,b,c,d\}}_2} = \omega(a,b) + \omega(a,c) + \omega(a,d) + \omega(b,c) + \omega(b,d) + \omega(c,d) + v^{\overline{\{a,b,c,d\}}_3}.$$

Thus $C_2(n) \subseteq W_2 + C_3(n)$. We show $W_2 \subseteq C_2(n)$. Let $\Delta = \{a,b,c,d,e,f\}$. Then

$$\sum_{x,y \in \Omega \setminus \Delta} \Big(v^{\overline{\{a,b,x,y\}}_2} + v^{\overline{\{a,b,f,x\}}_2} + v^{\overline{\{a,d,f,x\}}_2} + v^{\overline{\{a,c,e,x\}}_2} + v^{\overline{\{b,c,e,x\}}_2} + v^{\overline{\{d,c,e,x\}}_2} + v^{\overline{\{b,d,f,x\}}_2} \Big) + v^{\overline{\{c,d,e,f\}}_2} = \omega(a,b).$$

Therefore $C_3(n) \subseteq C_2(n)$, and $C_2(n) = W_2 + C_3(n)$. Now $\dim(C_2(n)) = \dim(W_2 + C_3(n)) \leq \binom{n-2}{3} + \binom{n}{2} - (n-1)$. Consider the independent vectors of $C_3(n)$ as in Theorem 2.7 and let $\Omega' = \Omega \setminus \{a, n-1, n\}$. Then for all $a \in \Omega \setminus \{n\}$,

$$\sum_{i,j \neq a,n} \omega(i,j) + \sum_{i,j,k \neq a,n} v^{\overline{\{i,j,k,n-1\}}_3} = \sum_{x,y \neq a,n} v^{\{a,x,y,n\}} = \omega(a,n).$$

Thus the $\omega(a,n)$'s are linearly dependent. Let a linear combination of the remaining vectors be zero:

$$\sum \alpha_{a,b}\omega(a,b) + \alpha_{c,d,e} v^{\overline{\{c,d,e,n-1\}}_3} = 0,$$

where $a,b \in \{1,2,\ldots,n-1\}$ and $c,d,e \in \{1,2,\ldots,n-2\}$. For $i,j,k \in \Omega \setminus \{n-1,n\}$, the coefficient of $v^{\{i,j,k,n\}}$ is $\alpha_{i,j} + \alpha_{i,k} + \alpha_{j,k} + \alpha_{i,j,k} = 0$. Hence for $i,j,k,l \in \Omega \setminus \{n-1,n\}$,

$$\alpha_{i,j,k} = \alpha_{i,j,l} + \alpha_{i,k,l} + \alpha_{j,k,l}. \tag{5}$$

The coefficient of each term is

$$\alpha_{i,j} + \alpha_{i,k} + \alpha_{i,n-1} + \alpha_{j,k} + \alpha_{j,n-1} + \alpha_{k,n-1} + \sum_{l \neq i,j,k,n-1,n} (\alpha_{i,j,l} + \alpha_{i,k,l} + \alpha_{j,k,l}) = 0.$$

By (5), $\alpha_{i,n-1} + \alpha_{j,n-1} + \alpha_{k,n-1} + (n-4)\alpha_{i,j,k} = \alpha_{i,n-1} + \alpha_{j,n-1} + \alpha_{k,n-1} = 0$ for $i \in \Omega \setminus \{n-1,n\}$. Thus all $\alpha_{i,n-1} = 0$. The coefficient of $v^{\{i,j,n-1,n\}}$ is

$$\alpha_{i,j} + \alpha_{i,n-1} + \alpha_{j,n-1} + \sum_{k \in \Omega \setminus \{i,j,n-1,n\}} \alpha_{i,j,k} = 0. \tag{6}$$

Adding the coefficients of $v^{\{i,j,n-1,n\}}$, $v^{\{i,k,n-1,n\}}$, and $v^{\{j,k,n-1,n\}}$ gives $\alpha_{i,j} + \alpha_{i,k} + \alpha_{j,k} + (n-2)\alpha_{i,j,k} = 0$. Since $\alpha_{i,j} + \alpha_{i,k} + \alpha_{j,k} + \alpha_{i,j,k} = 0$, all $\alpha_{i,j,k} = 0$. From (6), the remaining $\alpha_{i,j}$ are also zero. Therefore,

$$\dim(C_2(n)) = \dim\langle W_2 + C_3(n)\rangle = \binom{n-2}{3} + \binom{n}{2} - (n-1) = \binom{n}{3} - \binom{n-2}{2}.$$

□

3 Conclusions

The following table represents all of our results about binary codes from the uniform subset graph $G(n,4,i)$:

n	$4k$	$4k+1$	$4k+2$	$4k+3$
$C_0(n)$	for even k $F_2^{\mathcal{P}} = [\binom{n}{4}, \binom{n}{4}]_2$	$\left[\binom{n}{4}, \binom{n-1}{4}, 5\right]_2$	$[\binom{n}{4}, \binom{n}{4} - \binom{n}{2} + 1]_2$	$[\binom{n}{4}, \binom{n}{4} - \binom{n}{3} + 1]_2$
	for odd k $[\binom{n}{4}, \binom{n}{4} - 1, 2]_2$	$\left[\binom{n}{4}, \binom{n-1}{4} - 1\right]_2$	$[\binom{n}{4}, \binom{n}{4} - \binom{n}{2}]_2$	$[\binom{n}{4}, \binom{n}{4} - \binom{n}{3}]_2$
$C_1(n)$	$\left[\binom{n}{4}, \binom{n-2}{3}, 4(n-4)\right]_2$	$[\binom{n}{4}, \binom{n}{3} - \binom{n}{2}]_2$	$\left[\binom{n}{4}, \binom{n-1}{3} - \binom{n-1}{2}\right]_2$	$\left[\binom{n}{4}, \binom{n-1}{3}, n-3\right]_2$
$C_2(n)$			$\left[\binom{n}{4}, \binom{n}{3} - \binom{n-2}{2}\right]_2$	$[\binom{n}{4}, \binom{n}{3} - 1]_2$
$C_3(n)$	$\left[\binom{n}{4}, \binom{n-2}{3}, 4(n-4)\right]_2$	$\left[\binom{n}{4}, \binom{n-1}{3}, n-3\right]_2$	$\left[\binom{n}{4}, \binom{n-2}{3}, 4(n-4)\right]_2$	$\left[\binom{n}{4}, \binom{n-1}{3}, n-3\right]_2$

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Conflicts of Interests

The authors declare that they have no conflicts of interest regarding the publication of this article.

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