



Research Paper

Proof of a conjecture for the watching number

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Abstract. In this paper, we prove that for every connected graph G of order n , the watching number of its subdivision graph $S(G)$ is exactly n . This result confirms a conjecture previously proposed by Vatandoost et al. [7].

Keywords. watching system, subdivision, graph.

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1 Introduction

Throughout this paper, all graphs are finite, simple, and undirected. Let G be a graph. An edge with endpoints u and v is denoted by $u \sim v$. For every vertex x belong to $V(G)$, the open neighborhood is defined as $N_G(x) = \{y \in V(G) : x \sim y\}$, and the closed neighborhood is $N_G[x] = N_G(x) \cup \{x\}$. For a set $T \subseteq V(G)$, the open neighborhood of T is $N_G(T) = \cup_{x \in T} N_G(x)$ and the closed neighborhood of T is $N_G[T] = N_G(T) \cup T$. For a subset H of $V(G)$, the induced subgraph on H is denoted by $G[H]$. The graph $G \setminus H$ is obtained by removing the vertices in H and all edges incident to them.

A subset $S \subseteq V(G)$ is called a dominating set if every vertex of G belongs to S or has a neighbor

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in S . For a graph G , the *subdivision graph* $S(G)$ is obtained by inserting exactly one new vertex into each edge of G .

The concept of a *watching system* was proposed in [1] as an extension of identifying codes. For a graph G , a watcher is an ordered pair $\omega = (v_i, Z_i)$, where $v_i \in V(G)$ specifies the position of the watcher and $Z_i \subseteq N_G[v_i]$ represents the set of vertices monitored by it. We refer to Z_i as the *watching zone* of ω . A collection $W = \{\omega_1, \omega_2, \dots, \omega_k\}$ of watchers is called a *watching system* of G if $L_W(v) = \{\omega_i : v \in Z_i, 1 \leq i \leq k\}$ for distinct vertices are different and non-empty. The minimum cardinality of such a collection is called the *watching number* of G , denoted by $\omega(G)$. For further information, the reader can see [2, 8, 9].

Conjecture 1.1. *If G is a connected graph of order n , then $\omega(S(G)) = n$.*

At the end of this section, we highlight the main contributions of the paper. In a previous work [7], we proposed Conjecture 1.1 concerning the watching number of subdivision graphs $S(G)$. In the present paper, we settle this conjecture by providing a complete proof. Moreover, our argument shows that the result naturally extends to iterated subdivisions: for every integer $r \geq 1$, the watching number of $S^r(G)$ (obtained by subdividing each edge of G exactly r times) satisfies the same formula. Therefore, the conjecture is fully resolved for all iterated subdivision graphs.

2 Proof of the Conjecture

In this section, we show that Conjecture 1.1, is correct.

Theorem 2.1. [7] *Let G be a connected graph of order n . Then $\omega(S(G)) \leq n$.*

Theorem 2.2. *If G is a connected graph of order n , then $\omega(S(G)) = n$.*

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(S(G)) = V(G) \cup B$, where

$$B = \left\{ v_{ij} : 1 \leq i < j \leq n, N_{S(G)}(v_{ij}) = \{v_i, v_j\} \right\}.$$

By Theorem 2.1, $\omega(S(G)) \leq n$. On the contrary, assume that $\omega(S(G)) \leq n - 1$. Let W be a watching system of $S(G)$ with $|W| = \omega(S(G))$. We define

$$V_W = \{v \in V(S(G)) \mid (v, Z_v) \in W\},$$

the set of vertices of $S(G)$, where watchers are located. By the definition of watching system, we have $|V_W| \leq |W|$. If $V_W = V(G)$, then $n = |V(G)| = |V_W| \leq |W| \leq n - 1$, which is impossible. So $V_W \neq V(G)$.

We claim that

- i) $V_W \cap B \neq \emptyset$
- ii) $V_W \cap V(G) \neq \emptyset$.

Proof (i): Let $V_W \cap B = \emptyset$.

Since $V_W \neq V(G)$, there exists $v_0 \in V(G)$ such that $v_0 \notin V_W$. By the definition of watching system, V_W is a dominating set of $S(G)$. We know that $S(G)$ is a bipartite graph with two partitions $V(G)$ and B . It is easy to see that v_0 is not dominated by V_W , which is a contradiction. Hence $V_W \cap B \neq \emptyset$.

Proof (ii): Let $V_W \cap V(G) = \emptyset$.

If $V_W \neq B$, then there exists $b_0 \in B$ such that $b_0 \notin V_W$. Since $V_W \cap V(G) = \emptyset$, b_0 is not dominated by V_W , which is a contradiction. Thus $V_W = B$. Since $|E(G)| = |B|$ and $|V_W| \leq n - 1$, $|E(G)| \leq n - 1$. Thus G is a tree and $|E(G)| = n - 1$, because G is a connected graph. This show that there is exactly one watcher on vertices of B . Suppose that $v_i \in V(G)$ be a vertex of degree 1 in G and $v_j \in N_G(v_i)$. From watching system W , we imply that there exists watcher $\omega_{ij} = (v_{ij}, Z_{v_{ij}})$ such that $Z_{v_{ij}} = \{v_i, v_{ij}\}$ or $Z_{v_{ij}} = \{v_i, v_{ij}, v_j\}$. However, we have $L_W(v_i) = L_W(v_{ij}) = \{\omega_{ij}\}$. It contradicts with this fact that W is a watching system of $S(G)$. Therefore $V_W \cap V(G) \neq \emptyset$.

We define the set $\mathcal{A}$ as following

$$\mathcal{A} = \{|V_W \cap B| \mid W \text{ is a watching system of } S(G) \text{ such that } |W| = \omega(G)\}.$$

By Claim (i), $\min(\mathcal{A}) \geq 1$. Suppose that W_0 be a watching system of $S(G)$ with $|W| = \omega(S(G))$ such that $\min(\mathcal{A}) = |V_{W_0} \cap B|$. If $|V_{W_0} \cap B| = t$, then

$$|V_{W_0} \cap V(G)| \leq (n - 1) - t \quad \text{and} \quad |V(G) \setminus V_{W_0}| \geq t + 1.$$

First, we show that for every $v_{ij} \in V_{W_0} \cap B$ there is exactly one watcher on vertex v_{ij} . For the proof, suppose that there are two watchers $\omega_i = (v_{ij}, Z_i \subseteq N_{S(G)}(v_{ij}))$ and $\omega_j = (v_{ij}, Z_j \subseteq N_{S(G)}(v_{ij}))$ on vertex $v_{ij} \in V_{W_0} \cap B$. We define

$$\omega'_i = (v_i, \{v_i, v_{ij}\}), \quad \omega'_j = (v_j, \{v_j, v_{ij}\}) \quad \text{and} \quad W_1 = (W_0 \setminus \{\omega_i, \omega_j\}) \cup \{\omega'_i, \omega'_j\}.$$

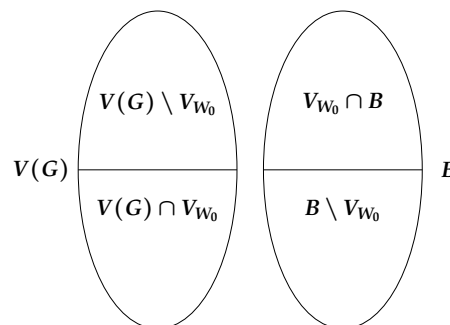
Then W_1 is a watching system of $S(G)$ with $\omega(S(G)) = |W_1|$ such that $|V_{W_1} \cap B| = |V_{W_0} \cap B| - 2$. Since $|V_{W_1} \cap B| \in \mathcal{A}$, $|V_{W_0} \cap B| \leq |V_{W_1} \cap B|$. Which is not true.

Let B_0 and B_1 be two subsets of $V_{W_0} \cap B$, where

$$B_0 = \{v_{ij} \in V_{W_0} \cap B : |\{v_i, v_j\} \cap V_{W_0}| = 0\},$$

and

$$B_1 = \{v_{ij} \in V_{W_0} \cap B : |\{v_i, v_j\} \cap V_{W_0}| = 1\}.$$



Suppose the induced subgraph on $(V_{W_0} \cap B) \cup (V(G) \setminus V_{W_0})$ be H .

Claim 1: $B_0 \neq \emptyset$.

For the proof, suppose that $B_0 = \emptyset$. By the definition of W_0 , for every $v \in V(G) \setminus V_{W_0}$, we have $\deg_H(v) \geq 1$. Since $|V(G) \setminus V_{W_0}| \geq t + 1$,

$$\sum_{v \in V(G) \setminus V_{W_0}} \deg_H(v) \geq t + 1.$$

It is clear that H is a bipartite graph. So,

$$\sum_{u \in V_{W_0} \cap B} \deg_H(u) = |E(H)| = \sum_{v \in V(G) \setminus V_{W_0}} \deg_H(v). \quad (*)$$

Since $B_0 = \emptyset$, we have $\sum_{u \in B_1} \deg_H(u) = |B_1| = \sum_{v \in V(G) \setminus V_{W_0}} \deg_H(v)$. Hence,

$$t \geq |V_{W_0} \cap B| \geq |B_1| = \sum_{u \in B_1} \deg_H(u) = \sum_{v \in V(G) \setminus V_{W_0}} \deg_H(v) \geq t + 1.$$

Which is a contradiction. Therefore, $B_0 \neq \emptyset$.

Claim 2: There exists at least a vertex $v \in N_H(B_0)$ with $\deg_H(v) = 1$. To prove the claim, let $\deg_H(v) \geq 2$, for all $v \in N_H(B_0)$. Then

$$\begin{aligned} \sum_{v \in V(G) \setminus V_{W_0}} \deg_H(v) &= \sum_{v \in N_H(B_1)} \deg_H(u) + \sum_{v \in N_H(B_0)} \deg_H(v) \\ &= |B_1| + \sum_{v \in N_H(B_0)} \deg_H(v) \geq |B_1| + 2((t + 1) - |B_1|). \end{aligned}$$

Also, we have $\sum_{u \in V_{W_0} \cap B} \deg_H(u) = 2t - |B_1|$. By $(*)$,

$$2(t + 1) - |B_1| \leq \sum_{v \in V(G) \setminus V_{W_0}} \deg_H(v) = \sum_{u \in V_{W_0} \cap B} \deg_H(u) = 2t - |B_1|.$$

It is impossible.

Now without loss of generality, we assume that $v_{ij} \in B_0$ such that $N_H(v_i) = \{v_{ij}\}$. Since there exists exactly one watcher on vertex $v_{ij} \in B_0$,

$$L_{W_0}(v_i) = L_{W_0}(v_{ij}) = \{\omega_{ij}\}.$$

It contradicts with this fact that W_0 is a watching system of $S(G)$.

Therefore $\omega(S(G)) = n$. □

Remark 2.3. We denote $S^0(G)$ by G . For $r \geq 1$, we write $S^r(G)$ for the subdivision graph $S^{r-1}(G)$. Clearly,

$$|V(S^r(G))| = n + (2^r - 1)m \quad \text{and} \quad |E(S^r(G))| = 2^r m,$$

where $|V(G)| = n$ and $|E(G)| = m$. Also, $S^r(G)$ is a connected graph if and only if G is a connected. Furthermore, $S^r(G)$ is a bipartite graph for every $r \geq 1$.

Corollary 2.4. If G is a graph of order n , then $\omega(S^r(G)) = n + (2^{r-1} - 1)m$.

Proof. By Remark 2.3 and Theorem 2.2, the result follows. □

3 Conclusion

In [11], it was conjectured that the watching number of the subdivision graph of G is equal to the order of G .

In this paper, we settle this conjecture affirmatively. More precisely, we have proved that if G is a connected graph of order n , then $\omega(S(G)) = n$.

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Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declare that they have no conflicts of interest regarding the publication of this article.

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