Journal of Discrete Mathematics and Its Applications 11 (3) (2026) 193–198

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Research Paper

Proof of a conjecture for the watching number

Ebrahim Vatandoost[1], Ahmad Abbasi[2], Kamran Mirasheh[2,*]

[1]Department of Basic Science, Imam Khomeini International University, P.O. Box 34148 - 96818, Qazvin, I. R. Iran

[2]Department of Pure Mathematics, Faculty of Mathematical Sciences, Guilan University, P. O. Box 41335-19141, Rasht, I. R. Iran

Academic Editor: Amir Hossein Ghodrati

Abstract. In this paper, we prove that for every connected graph G of order n, the watching number of its subdivision graph $S(G)$ is exactly n. This result confirms a conjecture previously proposed by Vatandoost et al. [7].

Keywords. watching system, subdivision, graph.
Mathematics Subject Classification (2020): 05C69.

1 Introduction

Throughout this paper, all graphs are finite, simple, and undirected. Let G be a graph. An edge with endpoints u and v is denoted by $u \sim v$. For every vertex x belonging to $V(G)$, the open neighborhood is defined as $N_G(x) = \{y \in V(G) : x \sim y\}$, and the closed neighborhood is $N_G[x] = N_G(x) \cup \{x\}$. For a set $T \subseteq V(G)$, the *open neighborhood* of T is $N_G(T) = \cup_{x \in T} N_G(x)$ and the closed neighborhood of T is $N_G[T] = N_G(T) \cup T$. For a subset H of $V(G)$, the induced subgraph on H is denoted by $G[H]$. The graph $G \setminus H$ is obtained by removing the vertices in H and all edges incident to them.
A subset $S \subseteq V(G)$ is called a dominating set if every vertex of G belongs to S or has a neighbor

*Corresponding author (Email address: k.mir.1185@gmail.com)
Received 22 October 2025; Revised 09 February 2026; Accepted 09 March 2026
First Publish Date: 01 September 2026

in S. For a graph G, the *subdivision graph* $S(G)$ is obtained by inserting exactly one new vertex into each edge of G.
The concept of a *watching system* was proposed in [1] as an extension of identifying codes. For a graph G, a watcher is an ordered pair $\omega = (v_i, Z_i)$, where $v_i \in V(G)$ specifies the position of the watcher and $Z_i \subseteq N_G[v_i]$ represents the set of vertices monitored by it. We refer to Z_i as the *watching zone* of ω. A collection $W = \{\omega_1, \omega_2, \ldots, \omega_k\}$ of watchers is called a *watching system* of G if $L_W(v) = \{\omega_i : v \in Z_i,\ 1 \leq i \leq k\}$ for distinct vertices are different and non-empty. The minimum cardinality of such a collection is called the *watching number* of G, denoted by $\omega(G)$. For further information, the reader can see [2,8,9].

Conjecture 1.1. *If G is a connected graph of order n, then $\omega(S(G)) = n$.*

At the end of this section, we highlight the main contributions of the paper. In a previous work [7], we proposed Conjecture 1.1 concerning the watching number of subdivision graphs $S(G)$. In the present paper, we settle this conjecture by providing a complete proof. Moreover, our argument shows that the result naturally extends to iterated subdivisions: for every integer $r \geq 1$, the watching number of $S^r(G)$ (obtained by subdividing each edge of G exactly r times) satisfies the same formula. Therefore, the conjecture is fully resolved for all iterated subdivision graphs.

2 Proof of the Conjecture

In this section, we show that Conjecture 1.1 is correct.

Theorem 2.1. *[7] Let G be a connected graph of order n. Then $\omega(S(G)) \leq n$.*

Theorem 2.2. *If G is a connected graph of order n, then $\omega(S(G)) = n$.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $V(S(G)) = V(G) \cup B$, where

$$B = \Big\{ v_{ij} \ :\ 1 \leq i < j \leq n,\ N_{S(G)}(v_{ij}) = \{v_i, v_j\} \Big\}.$$

By Theorem 2.1, $\omega(S(G)) \leq n$. On the contrary, assume that $\omega(S(G)) \leq n-1$. Let W be a watching system of $S(G)$ with $|W| = \omega(S(G))$. We define

$$V_W = \{v \in V(S(G)) \mid (v, Z_v) \in W\},$$

the set of vertices of $S(G)$, where watchers are located. By the definition of watching system, we have $|V_W| \leq |W|$. If $V_W = V(G)$, then $n = |V(G)| = |V_W| \leq |W| \leq n-1$, which is impossible. So $V_W \neq V(G)$.
We claim that

i) $V_W \cap B \neq \emptyset$

ii) $V_W \cap V(G) \neq \emptyset$.

Proof (i): Let $V_W \cap B = \emptyset$.
Since $V_W \neq V(G)$, there exists $v_0 \in V(G)$ such that $v_0 \notin V_W$. By the definition of watching system, V_W is a dominating set of $S(G)$. We know that $S(G)$ is a bipartite graph with two partitions $V(G)$ and B. It is easy to see that v_0 is not dominated by V_W, which is a contradiction. Hence $V_W \cap B \neq \emptyset$.
Proof (ii): Let $V_W \cap V(G) = \emptyset$.
If $V_W \neq B$, then there exists $b_0 \in B$ such that $b_0 \notin V_W$. Since $V_W \cap V(G) = \emptyset$, b_0 is not dominated by V_W, which is a contradiction. Thus $V_W = B$. Since $|E(G)| = |B|$ and $|V_W| \leq n-1$, $|E(G)| \leq n-1$. Thus G is a tree and $|E(G)| = n-1$, because G is a connected graph. This show that there is exactly one watcher on vertices of B. Suppose that $v_i \in V(G)$ be a vertex of degree 1 in G and $v_j \in N_G(v_i)$. From watching system W, we imply that there exists watcher $\omega_{ij} = (v_{ij}, Z_{v_{ij}})$ such that $Z_{v_{ij}} = \{v_i, v_{ij}\}$ or $Z_{v_{ij}} = \{v_i, v_{ij}, v_j\}$. However, we have $L_W(v_i) = L_W(v_{ij}) = \{\omega_{ij}\}$. It contradicts with this fact that W is a watching system of $S(G)$. Therefore $V_W \cap V(G) \neq \emptyset$.
We define the set $\mathcal{A}$ as follows

$$\mathcal{A} = \{|V_W \cap B| \mid W \textit{ is a watching system of } S(G) \textit{ such that } |W| = \omega(G)\}.$$

By Claim (i), $min(\mathcal{A}) \geq 1$. Suppose that W_0 is a watching system of $S(G)$ with $|W| = \omega(S(G))$ such that $min(\mathcal{A}) = |V_{W_0} \cap B|$. If $|V_{W_0} \cap B| = t$, then

$$|V_{W_0} \cap V(G)| \leq (n-1) - t \quad \textit{and} \quad |V(G) \setminus V_{W_0}| \geq t+1.$$

First, we show that for every $v_{ij} \in V_{W_0} \cap B$ there is exactly one watcher on vertex v_{ij}. For the proof, suppose that there are two watchers $\omega_i = (v_{ij}, Z_i \subseteq N_{S(G)}(v_{ij}))$ and $\omega_j = (v_{ij}, Z_j \subseteq N_{S(G)}(v_{ij}))$ on vertex $v_{ij} \in V_{W_0} \cap B$. We define

$$\omega_i' = (v_i, \{v_i, v_{ij}\}), \quad \omega_j' = (v_j, \{v_j, v_{ij}\}) \quad \textit{and} \quad W_1 = (W_0 \setminus \{\omega_i, \omega_j\}) \cup \{\omega_i', \omega_j'\}.$$

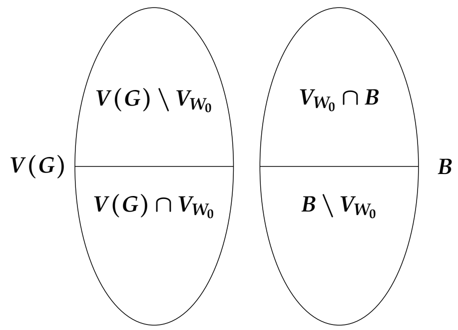
Then W_1 is a watching system of $S(G)$ with $\omega(S(G)) = |W_1|$ such that $|V_{W_1} \cap B| = |V_{W_0} \cap B| - 2$. Since $|V_{W_1} \cap B| \in \mathcal{A}$, $|V_{W_0} \cap B| \leq |V_{W_1} \cap B|$. Which is not true.

Let B_0 and B_1 be two subsets of $V_{W_0} \cap B$, where

$$B_0 = \{v_{ij} \in V_{W_0} \cap B : |\{v_i, v_j\} \cap V_{W_0}| = 0\},$$

and

$$B_1 = \{v_{ij} \in V_{W_0} \cap B : |\{v_i, v_j\} \cap V_{W_0}| = 1\}.$$



Suppose the induced subgraph on $(V_{W_0} \cap B) \cup (V(G) \setminus V_{W_0})$ be H.
Claim 1: $B_0 \neq \emptyset$.
For the proof, suppose that $B_0 = \emptyset$. By the definition of W_0, for every $v \in V(G) \setminus V_{W_0}$, we have $deg_H(v) \geq 1$. Since $|V(G) \setminus V_{W_0}| \geq t+1$,

$$\sum_{v \in V(G) \setminus V_{W_0}} deg_H(v) \geq t+1.$$

It is clear that H is a bipartite graph. So,

$$\sum_{u \in V_{W_0} \cap B} deg_H(u) = |E(H)| = \sum_{v \in V(G) \setminus V_{W_0}} deg_H(v). \quad (*)$$

Since $B_0 = \emptyset$, we have $\sum_{u \in B_1} deg_H(u) = |B_1| = \sum_{v \in V(G) \setminus V_{W_0}} deg_H(v)$. Hence,

$$t \geq |V_{W_0} \cap B| \geq |B_1| = \sum_{u \in B_1} deg_H(u) = \sum_{v \in V(G) \setminus V_{W_0}} deg_H(v) \geq t+1.$$

Which is a contradiction. Therefore, $B_0 \neq \emptyset$.
Claim 2: There exists at least a vertex $v \in N_H(B_0)$ with $\deg_H(v) = 1$. To prove the claim, let $deg_H(v) \geq 2$, for all $v \in N_H(B_0)$. Then

$$\begin{aligned}\sum_{v \in V(G) \setminus V_{W_0}} deg_H(v) &= \sum_{v \in N_H(B_1)} deg_H(u) + \sum_{v \in N_H(B_0)} deg_H(v) \\ &= |\, B_1| + \sum_{v \in N_H(B_0)} deg_H(v) \geq |B_1| + 2((t+1) - |B_1|).\end{aligned}$$

Also, we have $\sum_{u \in V_{W_0} \cap B} deg_H(u) = 2t - |\, B_1|$. By $(*)$,

$$2(t+1) - |B_1| \leq \sum_{v \in V(G) \setminus V_{W_0}} deg_H(v) = \sum_{u \in V_{W_0} \cap B} deg_H(u) = 2t - |\, B_1|.$$

It is impossible.
Now without loss of generality, we assume that $v_{ij} \in B_0$ such that $N_H(v_i) = \{v_{ij}\}$. Since there exists exactly one watcher on vertex $v_{ij} \in B_0$,

$$L_{W_0}(v_i) = L_{W_0}(v_{ij}) = \{\omega_{ij}\}.$$

It contradicts this fact that W_0 is a watching system of $S(G)$.
Therefore $\omega(S(G)) = n$.

□

Remark 2.3. *We denote $S^0(G)$ by G. For $r \geq 1$, we write $S^r(G)$ for the subdivision graph $S^{r-1}(G)$. Clearly,*

$$|V(S^r(G))| = n + (2^r - 1)m \quad \textit{and} \quad |E(S^r(G))| = 2^r m,$$

where $|V(G)| = n$ and $|E(G)| = m$. Also, $S^r(G)$ is a connected graph if and only if G is a connected. Furthermore, $S^r(G)$ is a bipartite graph for every $r \geq 1$.

Corollary 2.4. *If G is a graph of order n, then $\omega(S^r(G)) = n + (2^{r-1} - 1)m$.*

Proof. By Remark 2.3 and Theorem 2.2, the result follows.

□

3 Conclusion

In [11], it was conjectured that the watching number of the subdivision graph of G is equal to the order of G.
In this paper, we settle this conjecture affirmatively. More precisely, we have proved that if G is a connected graph of order n, then $\omega(S(G)) = n$.

Acknowledgment

The authors would like to express gratitude to the Reviewer for managing the concentrated target and duration required to review this manuscript.

Funding

This research received no external funding.

Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declare that they have no conflicts of interest regarding the publication of this article.

References

[1] D. Auger, I. Charon, O. Hudry, A. Lobstein, Watching systems in graphs: an extension of identifying codes, Discrete Applied Mathematics 161 (2013) 1674–1685. https://doi.org/10.1016/j.dam.2011.04.025
[2] D. Auger, I. Charon, O. Hudry, A. Lobstein, Maximum size of a minimum watching system and the graphs achieving the bound, Discrete Applied Mathematics 164 (2014) 20–33. https://doi.org/10.1016/j.dam.2012.08.028
[3] F. Foucaud, S. Gravier, R. Naserasr, A. Parreau, P. Valicov, Identifying codes in line graphs, Journal of Graph Theory 73 (2013) 425–448. https://doi.org/10.1002/jgt.21686
[4] F. Foucaud, G. Perarnau, Bounds for identifying codes in terms of degree parameters, Electronic Journal of Combinatorics 19 (2012) P32. https://doi.org/10.37236/2036
[5] M. D. Hernando, M. Mora, I. M. Pelayo, Watching systems in complete bipartite graphs, VIII Jornadas de Matemática Discreta y Algortímica, Almería (2012) 53–60.
[6] O. Hudry, A. Lobstein, Unique (optimal) solutions: Complexity results for identifying and locating dominating codes, Theoretical Computer Science 767 (2019) 83–102. https://doi.org/10.1016/j.tcs.2018.09.034
[7] K. Mirasheh, A. Abbasi, E. Vatandoost, Domination number and watching number of subdivision construction of graphs, Facta Universitatis, Series: Mathematics and Informatics 40 (2025) 33–43. https://doi.org/10.22190/FUMI221126003M
[8] M. Roozbayani, H. Maimani, A. Tehranian, Watching systems of triangular graphs, Transactions on Combinatorics 3 (2014) 51–57. https://doi.org/10.22108/toc.2014.4127

[9] M. Roozbayani, H. R. Maimani, Identifying codes and watching systems in Kneser graphs, Discrete Mathematics, Algorithms and Applications 9 (2017) 1750007. https://doi.org/10.1142/S1793830917500070

[10] D. F. Rall, K. Wash, Identifying codes of the direct product of two cliques, European Journal of Combinatorics 36 (2014) 159–171. https://doi.org/10.1016/j.ejc.2013.07.002

[11] A. Shaminejad, E. Vatandoost, K. Mirasheh, The identifying code number and Mycielski's construction of graphs, Transactions on Combinatorics 11 (2022) 309–316. https://doi.org/10.22108/toc.2021.126368.1794

Citation: E. Vatandoost, A. Abbasi, K. Mirasheh, Proof of a conjecture for the watching number, J. Disc. Math. Appl. 11(3) (2026) 193–198.

https://doi.org/10.22061/jdma.2026.12640.1177