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Research Paper

On mixed double Roman domination number and 2-independence number of graphs

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Abstract. Let $G = (V(G), E(G))$ be a simple graph. A mixed double Roman dominating function (MDRDF) on G is an assignment of labels from the set $\{0, 1, 2, 3\}$ to the vertices and edges of G such that every zero-labeled element is incident or adjacent to an element labeled 3 or incident or adjacent to at least two elements labeled 2, while every element labeled 1 is incident or adjacent to some element with label at least 2. The weight of such a labeling is the sum of all assigned labels, and the minimum possible weight over all MDRDFs of G is denoted by $\gamma^*_{dR}(G)$. A set $X \subseteq V(G)$ is called 2-independent if the induced subgraph $G[X]$ is isomorphic to $rK_2 \cup sK_1$ for some nonnegative integers r and s with $2r + s = |X|$. The maximum cardinality of such a set is denoted by $\beta_2(G)$. In this paper, we study the relationship between $\gamma^*_{dR}(G)$ and $\beta_2(G)$. We show that these two parameters are generally incomparable: there exist graphs in which either parameter can exceed the other by an arbitrarily large amount. For trees, however, we prove that if T belongs to a certain family $\mathcal{T}$, including all paths and trees containing a quasi-diagonal path $u_1, \ldots, u_k$ such that $d(u_2) \neq 2$ or $d(u_3) \neq 2$, then the inequality $\gamma^*_{dR}(T) \leq 2\beta_2(T)$ holds. Moreover, we fully characterize all trees in $\mathcal{T}$ for which equality is attained.

Keywords. 2-independence number, mixed double Roman domination number, tree.
Mathematics Subject Classification (2020): 05C69, 05C90, 05C05, 68R10.

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1 Introduction

Throughout this article, graphs are considered simple undirected. The *order* of a graph $G = (V(G), E(G))$ is the cardinality of $V(G)$ and its *size* is the cardinality of $E(G)$. For a vertex $v \in V(G)$, the *degree* of v, denoted by $d(v)$, is the number of edges incident to v. For a vertex $v \in V(G)$, we denote by $[v]$ the set consisting of v together with all edges incident to it. Thus, the degree of v satisfies $d(v) = |[v]| - 1$. For each element $x \in V(G) \cup E(G)$, the *neighborhood* of x is the set of all vertices and edges of G that are adjacent or incident to x. A vertex of degree 1 in a tree T is called a *leaf*. For an arbitrary vertex v in a rooted tree, let $D(v)$ denotes the set of descendants of v and the *depth* of v be the largest distance from v to a vertex in $D(v)$. The *diameter* of T, denoted by $\mathrm{diam}(T)$, is the greatest distance between two vertices of T. What we mean by *quasi-diagonal path* in a tree is a path like $P : u_1, \ldots, u_k$ such that both u_1 and u_k are leaves and there is no path with length greater than k that includes sub-path $u_3, \ldots, u_k$. We denote the *path* and the *star* of order n by P_n and $K_{1,n-1}$, respectively. A *double star* $DS_{p,q}$ is a tree obtained from $K_{1,p}$ and $K_{1,q}$ by connecting the center of $K_{1,p}$ with that of $K_{1,q}$. For functions $f : V(G) \to \mathbb{Z}$ and $g : V(G) \cup E(G) \to \mathbb{Z}$, we define $\omega(f) = \sum_{x \in V(G)} f(x)$ and $\omega(g) = \sum_{x \in V(G) \cup E(G)} g(x)$ and call them the *weight* of f and g, respectively.

As a pioneer, Cockayne et al [12] in 2004, Motivated by works [17, 24, 25], defined *Roman domination number* of G as a minimum weight among all *Roman dominating functions* on G (functions from $V(G)$ to $\{0, 1, 2\}$ such that every vertex with zero image under f is adjacent to at least one vertex with image 2 under f) and studied the graph theoretic properties of this variant of the domination number of G. To read more about the Roman domination number, see [7–10, 14] and the references therein.

Considering both edges and vertices in the above definition, first Abdollahzadeh Ahangar et al. [3] in 2015 and then Valinavaz [27] in 2022 defined mixed Roman domination and mixed double Roman domination numbers of a graph, respectively. A *mixed Roman dominating function* on G is a labeling of the vertices and edges by elements of $\{0, 1, 2\}$ such that every element labeled 0 is incident or adjacent or incident or adjacent to at least one element labeled 2. The minimum possible weight of such a labeling is the *mixed Roman domination number* of G. A *mixed double Roman dominating function* on G is a labeling of the vertices and edges by elements of $\{0, 1, 2, 3\}$ satisfying the condition that every element labeled 0 is incident to an element labeled 3 or adjacent to at least two elements labeled 2, while every element labeled 1 is adjacent or incident to at least one element with label at least 2. The minimum possible weight of such a labeling is the *mixed double Roman domination number* of G. For more information and details about the results stated on the Roman domination-like numbers that mentioned above, the reader can see [1, 2, 5, 6, 11, 16, 20–23, 26, 28–30].

One of the remarkable routines in the obtained results regarding the Roman domination type numbers is finding an upper bound for them in terms of other graph parameters of the given graph. A famous parameter that has attracted the attention of researchers is β_2

(see [4,13,18,19]). According to Fink and Jacobson [15], a subset X of $V(G)$ is 2-*independent* if the subgraph induced by the vertices of X is isomorphic to $rK_2 \cup sK_1$, where r and s are natural numbers with $2r + s = |X|$. The 2-*independence number* $\beta_2(G)$ is the maximum cardinality among all 2-independent sets of G. In this work, we show that the mixed double Roman domination number is absolutely incomparable with the 2-independence number. Next, we prove that for a large family of trees, the mixed double Roman domination number is less than or equal to 2 times the 2-independence number and we will also state the necessary and sufficient condition for the obtained boundary to be sharp.

2 Results

A *one-edge connection* of two graphs G_1 and G_2 is a graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{e = uv\}$, where $u \in V(G_1)$ and $v \in V(G_2)$ and is denoted by $G = G_1 \odot_{uv} G_2$. As a first result, with using of one-edge connection of graphs, we show that there exists a wide variety family of graphs that for each member G of this family, $\gamma^*_{dR}(G)$ is greater than $\beta_2(G)$ as much as we respect.

Proposition 2.1. *Let u be an arbitrary vertex of a graph G'. Then we have the following:*

1. $\beta_2(G) = \beta_2(G') + 2$ *and* $\gamma^*_{dR}(G') + 3 \leq \gamma^*_{dR}(G) \leq \gamma^*_{dR}(G') + 5$, *where* $G = G' \odot_{ux_3} P_3$;
2. $\beta_2(G) = \beta_2(G') + 3$ *and* $\gamma^*_{dR}(G') + 4 \leq \gamma^*_{dR}(G) \leq \gamma^*_{dR}(G') + 6$, *where* $G = G' \odot_{ux_3} P_4$;
3. $\beta_2(G) = \beta_2(G') + 4$ *and* $\gamma^*_{dR}(G') + 6 \leq \gamma^*_{dR}(G) \leq \gamma^*_{dR}(G') + 8$, *where* $G = G' \odot_{ux_3} P_5$.

Proof. (1) Clearly, any $\beta_2(G')$-set can be extended to a 2-independent set of G by adding x_1 and x_2. Hence, $\beta_2(G) \geq \beta_2(G') + 2$. Conversely, if S is a $\beta_2(G)$-set, then $|S \cap \{x_1, x_2, x_3\}| \leq 2$, and $S \cap V(G')$ is a 2-independent set of G'; therefore $\beta_2(G) \leq \beta_2(G') + 2$. Consequently, $\beta_2(G) = \beta_2(G') + 2$. Now, let f be a $\gamma^*_{dR}(G')$-function. Then the function $g : V(G) \cup E(G) \to \{0,1,2,3\}$ defined by $g(x_2) = 3$, $g(x_3u) = 2$, $g(x_1) = g(x_3) = g(x_1x_2) = g(x_2x_3) = 0$ and $g(x) = f(x)$ otherwise, is an MDRDF of G and so $\gamma^*_{dR}(G) \leq \gamma^*_{dR}(G') + 5$. Next, let f be a $\gamma^*_{dR}(G)$-function. One can easily check that $f(x_1) = f(x_3) = f(x_2x_3) = 0$, one of the two values of $f(x_1x_2)$ or $f(x_2)$ is zero and the other is 3 and $f(x_3u) \neq 1$. If $f(x_3u) = 0$, then $f\,|_{V(G') \cup E(G')}$ is an MDRDF of G' and so $\gamma^*_{dR}(G') \leq \gamma^*_{dR}(G) - 3$. Else, the function $g : V(G') \cup E(G') \to \{0,1,2,3\}$ defined by $g(u) = f(x_3u)$ and $g(x) = f(x)$ otherwise, is an MDRDF of G' and hence $\gamma^*_{dR}(G') \leq \gamma^*_{dR}(G) - 3$. Therefore, $\gamma^*_{dR}(G') + 3 \leq \gamma^*_{dR}(G) \leq \gamma^*_{dR}(G') + 5$.

(2) Similar to the method described in the proof of (1), the set obtained from the union of any $\beta_2(G')$-set and $\{x_1, x_2, x_4\}$ is a $\beta_2(G)$-set and hence $\beta_2(G) = \beta_2(G') + 3$. Now, let f be a $\gamma^*_{dR}(G')$-function. Then the function $g : V(G) \cup E(G) \to \{0,1,2,3\}$ defined by $g(x_2) = g(x_3) = 3$, $g(x_1) = g(x_4) = g(x_1x_2) = g(x_2x_3) = g(x_3x_4) = g(x_3u) = 0$ and $g(x) = f(x)$ otherwise, is an MDRDF of G and so $\gamma^*_{dR}(G) \leq \gamma^*_{dR}(G') + 6$. Next, suppose f is a $\gamma^*_{dR}(G)$-function. One can check that either $f(x_3u) = 0$ and $f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_1x_2) + f(x_2x_3) + f(x_3x_4) = 6$; or

$f(x_3u) \geq 2$ and $f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_1x_2)+f(x_2x_3)+f(x_3x_4) = 5$. If $f(x_3u) = 0$, then the function $g : V(G') \cup E(G') \to \{0,1,2,3\}$ defined by $g(u) = 2$ and $g(x) = f(x)$ otherwise, is an MDRDF of G' and hence $\gamma^*_{dR}(G') \leq \gamma^*_{dR}(G) - 4$. If $f(x_3u) \geq 2$, then the function $g : V(G') \cup E(G') \to \{0,1,2,3\}$ defined by $g(u) = 3$ and $g(x) = f(x)$ otherwise, is an MDRDF of G' and hence $\gamma^*_{dR}(G') \leq \gamma^*_{dR}(G) - 4$. Therefore, $\gamma^*_{dR}(G') + 4 \leq \gamma^*_{dR}(G) \leq \gamma^*_{dR}(G') + 6$.

(3) Clearly, the set consists of each $\beta_2(G')$-set and $\{x_1, x_2, x_4, x_5\}$ is a $\beta_2(G)$-set and so $\beta_2(G) = \beta_2(G') + 4$. Now, let f be a $\gamma^*_{dR}(G')$-function. Then the function $g : V(G) \cup E(G) \to \{0,1,2,3\}$ defined by $g(x_2) = g(x_4) = 3$, $g(x_3u) = 2$, $g(x_1) = g(x_3) = g(x_5) = g(x_1x_2) = g(x_2x_3) = g(x_3x_4) = g(x_4x_5) = 0$ and $g(x) = f(x)$ otherwise, is an MDRDF of G and so $\gamma^*_{dR}(G) \leq \gamma^*_{dR}(G') + 8$. Next, let f be a $\gamma^*_{dR}(G)$-function. One can check that $f(x_3) = 0$ and $f(x_1)+f(x_2)+f(x_4)+f(x_5)+f(x_1x_2)+f(x_2x_3)+f(x_3x_4)+f(x_4x_5) = 6$. Also, $f(x_3u) \neq 1$. If $f(x_3u) = 0$, then $f\,|_{G'}$ is an MDRDF of G' and hence $\gamma^*_{dR}(G') \leq \gamma^*_{dR}(G) - 6$. If $f(x_3u) \geq 2$, then the function $g : V(G') \cup E(G') \to \{0,1,2,3\}$ defined by $g(u) = f(x_3u)$ and $g(x) = f(x)$ otherwise, is an MDRDF of G' and hence $\gamma^*_{dR}(G') \leq \gamma^*_{dR}(G) - 6$. Therefore, $\gamma^*_{dR}(G') + 6 \leq \gamma^*_{dR}(G) \leq \gamma^*_{dR}(G') + 8$. □

With the help of Proposition 2.1 and using the pathes P_3, P_4 and P_5, many families of graphs can be made such that the value of $\gamma^*_{dR} - \beta_2$ for each of the graphs in these families can be as large as we respect.

In the following example, we introduce a family of graphs that the opposite of the above-mentioned statement is true about them.

Example 2.2. *(i) Let p be a natural number. Then $\beta_2(K_{1,p+3}) = p + 3$ and $\gamma^*_{dR}(K_{1,p+3}) = 3$. Thus $\beta_2(K_{1,p+3}) - \gamma^*_{dR}(K_{1,p+3}) = p$.*
(ii) Let r be a natural number and $p_1, \ldots, p_s$ ($s \leq r$) be positive integers. Suppose v_i's be leaves of the star graph $K_{1,r}$. Consider $K_{1,r}$ and for each $i = 1, \ldots, s$ identify each v_i with the center of K_{1,p_i}. We denote this generalized star graph by $GS_{r,p_1,\ldots,p_s}$ (see Fig. 1). The function $f : V(GS_{r,p_1,\ldots,p_s}) \cup E(GS_{r,p_1,\ldots,p_s}) \to \{0,1,2,3\}$ defined by $f(v_i) = 3$ for any v_i and $f(x) = 0$ otherwise, is an MDRDF of $GS_{r,p_1,\ldots,p_s}$ and hence $\gamma^*_{dR}(GS_{r,p_1,\ldots,p_s}) \leq 3r$. On the other hand, $\beta_2(GS_{r,p_1,\ldots,p_s}) \geq p_1 + \cdots + p_s + r - s$. Therefore, given sufficiently large p_i's, we can see that $\beta_2(GS_{r,p_1,\ldots,p_s}) - \gamma^*_{dR}(GS_{r,p_1,\ldots,p_s})$ becomes larger than any number we respect.

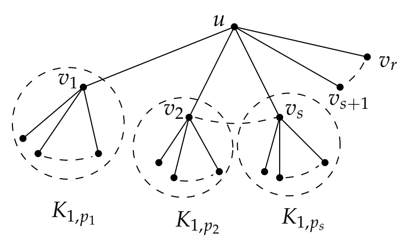

Figure 1. The graph $GS_{r,p_1,\ldots,p_s}$.

It was observed that for an arbitrary graph G, the mixed double Roman domination number

$\gamma^*_{dR}(G)$ and the 2-independence number $\beta_2(G)$ are not comparable with each other, even if G is a tree. But in the following, we express the upper bound for γ^*_{dR} in a wide variety family of trees in terms of β_2. Suppose that $\mathcal{T}$ is a family consisting of paths P_n for all n and trees T with at least a quasi-diagonal path $u_1,\ldots,u_k$ with $d(u_2)\neq 2$ or $d(u_3)\neq 2$.

Proposition 2.3. *For a natural number $n\geq 2$, we have*

$$\gamma^*_{dR}(P_n)=\begin{cases}6k & n=5k,5k-1\\ 6k+2 & n=5k+1\\ 6k+3 & n=5k+2,5k+3\end{cases}, \quad \beta_2(P_n)=\begin{cases}2k & n=3k,3k-1\\ 2k+1 & n=3k+1\end{cases}$$

Proof. According to Proposition 1 of [28], $\gamma^*_{dR}(P_n)=\lceil\frac{6n-3}{5}\rceil$ if $n\overset{5}{\equiv}0,3$ and $\gamma^*_{dR}(P_n)=\lceil\frac{6n-3}{5}\rceil+1$ if $n\overset{5}{\equiv}1,2,4$. Also, if $P_n: v_1,v_2,\ldots,v_n$, then by selecting all vertices except those with indices multiple of 3, we have a maximal 2-independent set and the result follows. $\square$

According to Proposition 2.3, clearly $\gamma^*_{dR}(P_n)\leq 2\beta_2(P_n)$ for each $n\geq 2$ and equality occurs if and only if $n=4,6,9$.

Theorem 2.4. *For any $T\in\mathcal{T}$, $\gamma^*_{dR}(T)\leq 2\beta_2(T)$.*

Proof. Let T be a tree with m edges. We proceed by induction on m. For $m=1$, we have $T=P_2$, and $\gamma^*_{dR}(T)=3<4=2\beta_2(T)$. Now, suppose that $m\geq 2$ and the result is valid for all trees of size less than or equal to $m-1$ and T is a tree of size m. If $\mathrm{diam}(T)=2$, then $T\cong K_{1,m}$. So $\gamma^*_{dR}(K_{1,m})=3<2m=2\beta_2(K_{1,m})$. Next, suppose $\mathrm{diam}(T)=3$. So, $T\cong DS_{p,q}$. Without loss of generality, suppose $p\geq q$. Thus

$$\gamma^*_{dR}(DS_{p,q})=6\leq\begin{cases}2(p+q) & q\geq 2\\ \\ 2(p+q+1) & \text{otherwise}\end{cases}=2\beta_2(DS_{p,q}).$$

Also, note $\gamma^*_{dR}(DS_{p,q})=2\beta_2(DS_{p,q})$ if and only if $p=q=1$ or equivalently $DS_{p,q}=P_4$. Now, let $\mathrm{diam}(T)\geq 4$, $T\neq P_n$ (since the theorem is true for P_n, by Proposition 2.3) and $P=u_1\cdots u_k$ be a quasi-diagonal path of T in which u_2 has the largest degree among all the quasi-diagonal paths of length $k=2,\ldots,\mathrm{diam}(T)+1$. Root T at u_k. We consider the following cases.

Case 1. Let $d(u_2)=t\geq 3$. Suppose T' is the connected component of T resulting from removing u_2u_3 which contains u_3. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding $t-1$ leaves of u_2. Also, any $\gamma^*_{dR}(T')$-function can be extended to an MDRDF of T by assigning 3 to u_2 and 0 to its neighbors. Thus by the induction hypothesis we have,

$$\gamma^*_{dR}(T)\leq\gamma^*_{dR}(T')+3\leq 2(\beta_2(T)-(t-1))+3=2\beta_2(T)-2t+5<2\beta_2(T).$$

Case 2. Let $d(u_2)=2$. So one can assume that $d(u_3)\geq 3$, since $T\in\mathcal{T}$. Suppose u_3 has t children $v_1=u_2,v_2,\ldots,v_t$ with depth 1 and degree 2 and ℓ children with depth 0.

Subcase 2.1. Let $\ell = 0$ and T' be the connected component of T resulting from removing u_3u_4 which contains u_4. First note that $t \geq 2$. Also, one can easily see that $\beta_2(T) \geq \beta_2(T') + 2t$. On the other hand, any $\gamma^*_{dR}(T')$-function can be extended to an MDRDF of T by assigning 2 to u_3u_4, 3 to v_i's and 0 to remaining elements of $V(T) \cup E(T)$. Therefore:

$$\gamma^*_{dR}(T) \leq \gamma^*_{dR}(T') + 3t + 2.$$

It follows from the induction hypothesis that,

$$\gamma^*_{dR}(T) \leq 2\beta_2(T) - 4t + 3t + 2 = 2\beta_2(T) - t + 2 \leq 2\beta_2(T).$$

Further if $\gamma^*_{dR}(T) = 2\beta_2(T)$, then $t = 2$ and so $T \cong T' \odot_{u_4u_3} P_5$.

Subcase 2.2. Suppose $\ell \geq 1$ and T' be the connected component of T which was stated in Subcase 2.1. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding v_i's and $\ell + t$ leaves of T that adjacent to u_3 and v_i's. So $\beta_2(T) \geq \beta_2(T') + \ell + 2t$. Also, each $\gamma^*_{dR}(T')$-function can be extended to an MDRDF of T by assigning 3 to u_3 and v_i's; and 0 to remaining elements. It follows from the induction hypothesis that,

$$\gamma^*_{dR}(T) \leq \gamma^*_{dR}(T') + 3t + 3 \leq 2\beta_2(T) - 2\ell - 4t + 3t + 3 = 2\beta_2(T) + 3 - (2\ell + t) \leq 2\beta_2(T).$$

Moreover, if $\gamma^*_{dR}(T) = 2\beta_2(T)$, then $\ell = t = 1$ and so $T \cong T' \odot_{u_4u_3} P_4$. □

Let u be an arbitrary vertex of G. A function $f : V(G) \cup E(G) \to \{0,1,2,3\}$ is said to be an *almost mixed double Roman dominating function* ($AMDRDF$) with respect to u, if each element in $V(G) \cup E(G) \setminus \{[u]\}$ is mixed double Roman dominated under f. We define:

$$\gamma^*_{dR}(G;u) := \min\{\omega(f) \mid f \text{ is an AMDRDF with respect to } u\}.$$

Clearly, any MDRDF is an AMDRDF with respect to u and hence the above set is non-empty. Therefore, $\gamma^*_{dR}(G;u)$ is well defined and $\gamma^*_{dR}(G;u) \leq \gamma^*_{dR}(G)$. Also, if f is an AMDRDF with respect to u, then $g : V(G) \cup E(G) \to \{0,1,2,3\}$ with $g(u) = 3$ and $g(x) = f(x)$ otherwise is an MDRDF and so $\gamma^*_{dR}(G) \leq \gamma^*_{dR}(G;u) + 3$. Hence we have:

$$\gamma^*_{dR}(G) - 3 \leq \gamma^*_{dR}(G;u) \leq \gamma^*_{dR}(G).$$

In Figure 2, some examples for all possible states can be seen. In Proposition 2.1, we proved that for an arbitrary tree T, $\gamma^*_{dR}(T \odot_{ux_3} P_4) \leq \gamma^*_{dR}(T) + 6$ and $\gamma^*_{dR}(T \odot_{ux_3} P_5) \leq \gamma^*_{dR}(T) + 6$. In the next two theorems, we show for which trees the aforementioned inequalities will be sharp. To better understand the proofs, examples from Figure 2 are used as appropriate (see Figures 3 to 10).

Proposition 2.5. *Let u be an arbitrary vertex of a tree T' and $T = T' \odot_{ux_3} P_4$. Then $\gamma^*_{dR}(T) = \gamma^*_{dR}(T') + 6$ if and only if $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u)$ or $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) + 1$ or $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) + 2$ and for any $\gamma^*_{dR}(T';u)$-function f there exists some edge e in $[u]$ with $f(e) = 0$.*

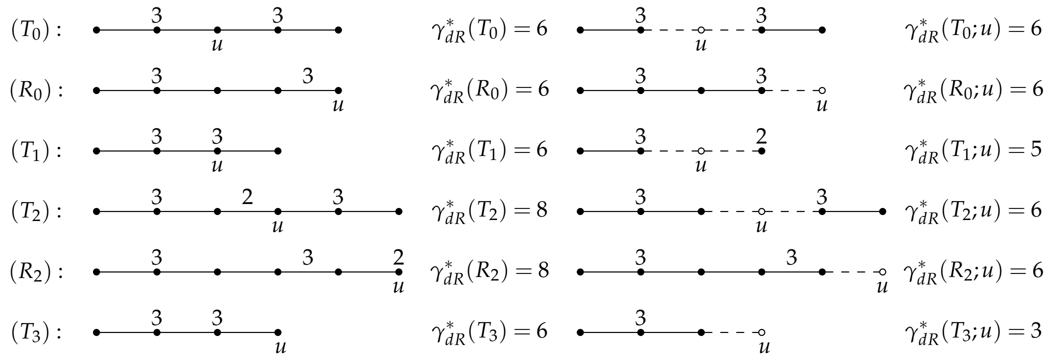

Figure 2. The graphs T_i and R_j $\quad i = 0,1,2,3$ and $j = 0,2$.

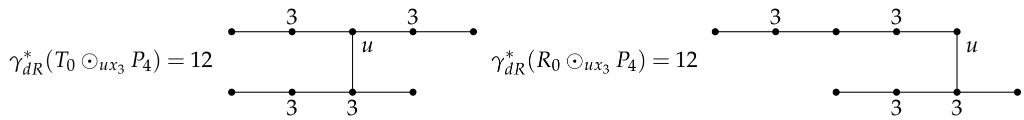

Figure 3. The graphs $T_0 \odot_{ux_3} P_4$ and $R_0 \odot_{ux_3} P_4$.

Proof. Let f be a $\gamma^*_{dR}(T)$-function. Clearly $\ell = f(x_1) + f(x_1x_2) + f(x_2) + f(x_2x_3) + f(x_3) + f(x_3x_4) + f(x_4) + f(x_3u) \geq 6$. Also, $f|_{V(T')\cup E(T')}$ is an AMDRDF. Now, we consider the following four cases:

Case 1. Let $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u)$. So $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) \leq \omega(f|_{V(T')\cup E(T')}) \leq \gamma^*_{dR}(T) - 6$ and hence $\gamma^*_{dR}(T) = \gamma^*_{dR}(T') + 6$, by Proposition 2.1.

Case 2. Let $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) + 1$. We have two following subcases:

Subcase 2.1. If $\ell \geq 7$, then $\gamma^*_{dR}(T') - 1 = \gamma^*_{dR}(T';u) \leq \omega(f|_{V(T')\cup E(T')}) \leq \gamma^*_{dR}(T) - 7$. Therefore $\gamma^*_{dR}(T) \geq \gamma^*_{dR}(T') + 6$ and consequently $\gamma^*_{dR}(T) = \gamma^*_{dR}(T') + 6$, by Proposition 2.1.

Subcase 2.2. If $\ell = 6$, then $f(x_3) = 3$ and $f(x_3u) = 0$. Also, $\omega(f|_{V(T')\cup E(T')}) \geq \gamma^*_{dR}(T';u) = \gamma^*_{dR}(T') - 1$. If $\omega(f|_{V(T')\cup E(T')}) = \gamma^*_{dR}(T') - 1$, then $f|_{V(T')\cup E(T')}$ is a $\gamma^*_{dR}(T';u)$-function. But we know that in the case $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) + 1$, all elements of $[u]$ are not dominated in any $\gamma^*_{dR}(T';u)$-function and the number assigned to only one of the vertices adjacent to u in any $\gamma^*_{dR}(T';u)$-function should be equal to 2 (we call this vertex v) and the number assigned to the other vertices adjacent to u and edges in $[u]$ should be zero, which with changing 2 to zero and assigning 3 to u, we obtain a $\gamma^*_{dR}(T')$-function. This implies that the edge uv is not dominated in f, a contradiction. Therefore, $\omega(f|_{V(T')\cup E(T')}) \geq \gamma^*_{dR}(T')$ and so $\gamma^*_{dR}(T) - 6 \geq \gamma^*_{dR}(T')$. Hence, $\gamma^*_{dR}(T) = \gamma^*_{dR}(T') + 6$, by Proposition 2.1.

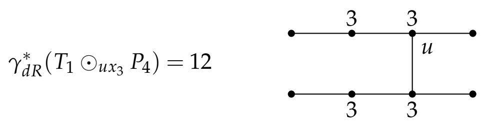

Figure 4. The graph $T_1 \odot_{ux_3} P_4$.

Case 3. Let $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) + 2$. In this case, for each $\gamma^*_{dR}(T';u)$-function, only one member of $[u]$ is not dominated and by assigning 2 to this element we obtain a $\gamma^*_{dR}(T')$-function. Consider two following subcases:

Subcase 3.1. Suppose g is a $\gamma^*_{dR}(T';u)$-function and u is not dominated. We can extend g to an MDRDF of T by assigning 3 to x_2 and x_3 and 0 to the other elements of $V(T) \cup E(T)$. Thus $\gamma^*_{dR}(T) \leq (\gamma^*_{dR}(T') - 2) + 6 = \gamma^*_{dR}(T') + 4$.

Subcase 3.2. Suppose for any $\gamma^*_{dR}(T';u)$-function there exists some edge e in $[u]$ that is not dominated.

Subcase 3.2.1. If $\ell \geq 8$, then $\gamma^*_{dR}(T') - 2 = \gamma^*_{dR}(T';u) \leq \omega(f|_{V(T') \cup E(T')}) \leq \gamma^*_{dR}(T) - 8$. Therefore $\gamma^*_{dR}(T) \geq \gamma^*_{dR}(T') + 6$ and consequently $\gamma^*_{dR}(T) = \gamma^*_{dR}(T') + 6$, by Proposition 2.1.

Subcase 3.2.2. If $\ell = 7$, then $f(x_3) = 0$ and $f(x_3u) = f(x_4) = 2$. Also, $\omega(f|_{V(T') \cup E(T')}) \geq \gamma^*_{dR}(T') - 2$. If $\omega(f|_{V(T') \cup E(T')}) = \gamma^*_{dR}(T') - 2$, then $f|_{V(T') \cup E(T')}$ is a $\gamma^*_{dR}(T';u)$-function. Therefore, e is adjacent to an element with assigned number 2 in T' and hence by assigning 1 to e, a $\gamma^*_{dR}(T')$-function is obtained. As a result, $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) + 1$ and this is a contradiction. Thus $\omega(f|_{V(T') \cup E(T')}) \geq \gamma^*_{dR}(T') - 1$ and so $\gamma^*_{dR}(T) - 7 \geq \gamma^*_{dR}(T') - 1$. Hence $\gamma^*_{dR}(T) \geq \gamma^*_{dR}(T') + 6$ and consequently $\gamma^*_{dR}(T) = \gamma^*_{dR}(T') + 6$, by Proposition 2.1.

Subcase 3.2.3. If $\ell = 6$, then $f(x_3) = 3$ and $f(x_3u) = 0$. Also, $\omega(f|_{V(T') \cup E(T')}) \geq \gamma^*_{dR}(T') - 2$. With a similar argument in Subcase 3.2.2 one can check that $\omega(f|_{V(T') \cup E(T')}) \geq \gamma^*_{dR}(T')$. So $\gamma^*_{dR}(T) \geq \gamma^*_{dR}(T') + 6$ and consequently $\gamma^*_{dR}(T) = \gamma^*_{dR}(T') + 6$, by Proposition 2.1.

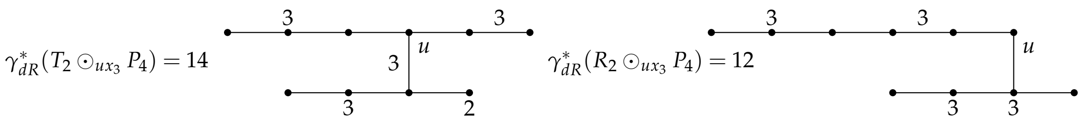

Figure 5. The graphs $T_2 \odot_{ux_3} P_4$ and $R_2 \odot_{ux_3} P_4$.

Case 4. Let $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) + 3$ and g' be a $\gamma^*_{dR}(T';u)$-function. So $\omega(g') = \gamma^*_{dR}(T') - 3$. The function $g : V(T) \cup E(T) \to \{0,1,2,3\}$ defined by $g(x_2) = g(x_3u) = 3$, $g(x_4) = 2$, $g(x_1) = g(x_1x_2) = g(x_2x_3) = g(x_3) = g(x_3x_4) = 0$ and $g(x) = g'(x)$ otherwise, is an MDRDF of T with $\omega(g) = \gamma^*_{dR}(T') + 5$. Hence $\gamma^*_{dR}(T) \leq \gamma^*_{dR}(T') + 5$.

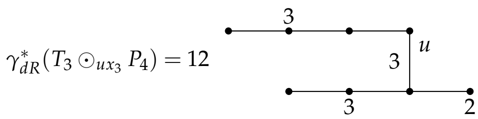

Figure 6. The graph $T_3 \odot_{ux_3} P_4$.

According to the equalities and inequalities obtained in the above four cases, the result follows. □

Proposition 2.6. *Let u be an arbitrary vertex of a tree T' and $T = T' \odot_{ux_3} P_5$. Then $\gamma^*_{dR}(T) = \gamma^*_{dR}(T') + 8$ if and only if $\gamma^*_{dR}(T';u) = \gamma^*_{dR}(T')$ and $f([u]) = 0$ for any $\gamma^*_{dR}(T';u)$-function f.*

Proof. We prove the proposition in the following four cases:

Case 1. Let $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u)$. First note that if g' is a $\gamma^*_{dR}(T')$-function with $g(e) =$

$2(g(e)=3)$ for some edge $e \in [u]$, then the function $g : V(T) \cup E(T) \to \{0,1,2,3\}$ defined by $g(x_2) = g(x_4) = 3$, $g(x_3u) = 1(g(x_3u) = 0)$, $g(x_1) = g(x_1x_2) = g(x_2x_3) = g(x_3) = g(x_3x_4) = g(x_4x_5) = g(x_5) = 0$ and $g(x) = g'(x)$ otherwise, is an MDRDF of T with $\omega(g) = \gamma^*_{dR}(T') + 7$ $(\omega(g) = \gamma^*_{dR}(T') + 6)$. Hence $\gamma^*_{dR}(T) \le \gamma^*_{dR}(T') + 7$. Next, we assume that $g'([u]) = 0$ for any $\gamma^*_{dR}(T')$-function g' and f is a $\gamma^*_{dR}(T)$-function. Clearly it can be assumed that $f(x_1) + f(x_1x_2) + f(x_2) + f(x_2x_3) + f(x_3) + f(x_3x_4) + f(x_4) + f(x_4x_5) + f(x_5) = 6$. Also, $f|_{V(T')\cup E(T')}$ is an AMDRDF. We have two following subcases:

Subcase 1.1. If $f(x_3u) \ge 2$, then $\omega(f|_{V(T')\cup E(T')}) \le \gamma^*_{dR}(T) - 8$ and so $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) \le \omega(f|_{V(T')\cup E(T')}) \le \gamma^*_{dR}(T) - 8$. Thus $\gamma^*_{dR}(T') \le \gamma^*_{dR}(T) - 8$ and hence $\gamma^*_{dR}(T) = \gamma^*_{dR}(T') + 8$, by Proposition 2.1.

Subcase 1.2. If $f(x_3u) = 0$, then $f_{V(T')\cup E(T')}$ is a $\gamma^*_{dR}(T')$-function. Therefore, $f([u]) = 0$ and consequently x_3u is not dominated, a contradiction.

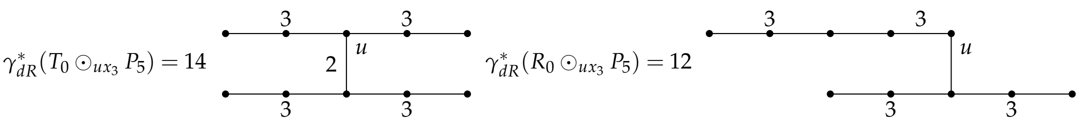

Figure 7. The graphs $T_0 \odot_{ux_3} P_5$ and $R_0 \odot_{ux_3} P_5$.

Case 2. Let $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) + 1$. Based on the explanations given in the proof process of Case 2 of Theorem 2.5, there exists a $\gamma^*_{dR}(T')$-function g' with $g'(u) = 3$. So g' can be extended to an MDRDF of T by assigning 3 to x_2 and x_4 and 0 to the other remaining members of $V(T) \cup E(T)$. Thus $\gamma^*_{dR}(T) \le \gamma^*_{dR}(T') + 6$.

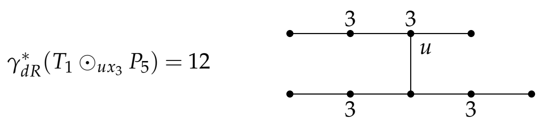

Figure 8. The graph $T_1 \odot_{ux_3} P_5$.

Case 3. Let $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) + 2$ and g' be a $\gamma^*_{dR}(T';u)$-function. So $\omega(g') = \gamma^*_{dR}(T') - 2$. The function $g : V(T) \cup E(T) \to \{0,1,2,3\}$ defined by $g(x_2) = g(x_3u) = g(x_4) = 3$, $g(x_1) = g(x_1x_2) = g(x_2x_3) = g(x_3) = g(x_3x_4) = g(x_4x_5) = g(x_5) = 0$ and $g(x) = g'(x)$ otherwise, is an MDRDF of T with $\omega(g) = (\gamma^*_{dR}(T') - 2) + 9$. Hence $\gamma^*_{dR}(T) \le \gamma^*_{dR}(T') + 7$.

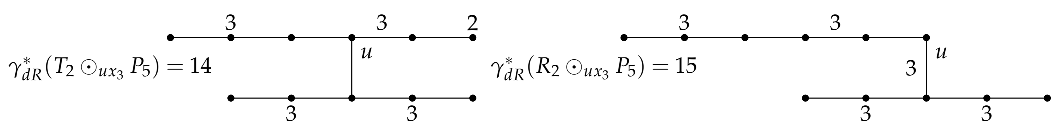

Figure 9. The graphs $T_2 \odot_{ux_3} P_5$ and $R_2 \odot_{ux_3} P_5$.

Case 4. Let $\gamma^*_{dR}(T') = \gamma^*_{dR}(T';u) + 3$. Similar to the argument mentioned in Case 3, one can see that there exists an MDRDF of T with weight $(\gamma^*_{dR}(T')-3)+9$ and so $\gamma^*_{dR}(T) \leq \gamma^*_{dR}(T')+6$.

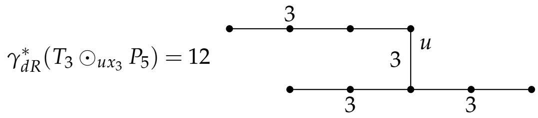

Figure 10. The graph $T_3 \odot_{ux_3} P_5$.

According to the equalities and inequalities obtained in the above four cases, the result follows. □

Now, let $\mathcal{T}' \subseteq \mathcal{T}$ be the family of unlabeled trees T for which there exists a sequence of trees $T_1, T_2, \ldots, T_k$ such that T_1 is one of the paths P_4, P_6 or P_9 and T_i can be obtained recursively from T_{i-1} by one of the following operations for $2 \leq i \leq k$.
Operation †. T_i is constructed by one-edge connecting T_{i-1} and P_4 through vertices $u \in V(T_{i-1})$ and $x_3 \in V(P_4)$, where u satisfies one of the following conditions:

($\dagger_0$) $\gamma^*_{dR}(T_{i-1}; u) = \gamma^*_{dR}(T_{i-1})$.

($\dagger_1$) $\gamma^*_{dR}(T_{i-1}; u) = \gamma^*_{dR}(T_{i-1}) + 1$.

($\dagger_2$) $\gamma^*_{dR}(T_{i-1}; u) = \gamma^*_{dR}(T_{i-1}) + 2$ and for any $\gamma^*_{dR}(T'; u)$-function f there exists some edge $e \in [u]$ with $f(e) = 0$.

Operation ‡. T_i is constructed by one-edge connecting T_{i-1} and P_5 through vertices $u \in V(T_{i-1})$ and $x_3 \in V(P_5)$, where $\gamma^*_{dR}(T_{i-1}; u) = \gamma^*_{dR}(T_{i-1})$ and $f([u]) = 0$ for any $\gamma^*_{dR}(T_{i-1}; u)$-function f.

Theorem 2.7. *Let $T \in \mathcal{T}'$ be an arbitrary tree. Then $\gamma^*_{dR}(T) = 2\beta_2(T)$.*

Proof. The proof is clear by Propositions 2.1, 2.3, 2.5 and 2.6 and Theorem 2.4. □

From the contents mentioned up to this point of the article, it can be seen that if $T = T' \odot_{ux_3} P_3$, then $\gamma^*_{dR}(T) = \gamma^*_{dR}(T') + 5$ if and only if $\gamma^*_{dR}(T'; u) = \gamma^*_{dR}(T')$ and $f([u]) = 0$ for any $\gamma^*_{dR}(T'; u)$-function f. Therefore, in the rest of the cases, if $\gamma^*_{dR}(T') \leq 2\beta_2(T')$, then $\gamma^*_{dR}(T) \leq \gamma^*_{dR}(T') + 4 = 2(\beta_2(T') + 2) = 2\beta_2(T)$. Our guess is that in the mentioned case, $\gamma^*_{dR}(T') < 2\beta_2(T')$. So we finish this paper by proposing the following conjecture and question.
Conjecture. Let T be an arbitrary tree. Then $\gamma^*_{dR}(T) \leq 2\beta_2(T)$.
Question. Which is the necessary and sufficient condition for the inequality of the previous conjecture to be sharp?

3 Conclusion

In this paper, we investigated the relationship between the mixed double Roman domination number $\gamma^*_{dR}(G)$ and the 2-independence number $\beta_2(G)$ of a graph G. We first showed that these two parameters are generally incomparable by constructing families of graphs where the difference $\gamma^*_{dR} - \beta_2$ can be arbitrarily large or arbitrarily small.

For trees, however, we established that the inequality $\gamma^*_{dR}(T) \leq 2\beta_2(T)$ holds for a broad family $\mathcal{T}$ of trees, which includes all paths and all trees containing a quasi-diagonal path $u_1, \ldots, u_k$ with $d(u_2) \neq 2$ or $d(u_3) \neq 2$. The proof is based on a structural induction and a careful analysis of the local configurations around the support vertices. We also characterized the extremal trees in $\mathcal{T}$ for which equality holds, showing that they can be built from small paths P_4, P_6 or P_9 by repeated one-edge connecting P_4 or P_5 under certain conditions involving the almost mixed double Roman domination number.

Our results suggest that the inequality may hold for all trees, which we stated as a conjecture. We also left open the question of a full characterization of trees attaining equality in the bound.

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Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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