Journal of Discrete Mathematics and Its Applications 11 (2) (2026) 123–129

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Research Paper

A walk on MLDR and MHDR codes

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Academic Editor: Hamidreza Maimani

Abstract. For a code D of length l over $\mathbb{Z}_4$, we denote by $M(D)$ the matrix containing all code words of D on its rows. Any column of $M(D)$ corresponds to the column which is zero or it has zero and 2 equally or it has all elements of $\mathbb{Z}_4$ equally. The Lee weight for these columns is defined 0, 2 and 1, respectively. If we calculate the sum of all Lee weights of columns of $M(D)$, it is denoted by $wt_L(D)$ and called the Lee support weight of D. In addition, The $m-$th Generalized Lee weight (GLW) for D, denoted by $d_m^L(D)$, is defined as the minimum of the Lee support weights of all submodules of D of Rank m. In the other words,

$$d_m^L(D) = min\{wt_L(E); E \text{ is a } \mathbb{Z}_4 - \text{submodule of D , rank(E)} = m\}.$$

It is obtained that for $m, 1 \leq m \leq rank(D)$,We have

$$\lfloor \frac{d_m^L(D) - 2m + 1}{2} \rfloor \leq l - rank(D).$$

The code which meets the recent upper bound is called maximum Lee distance separable with respect to rank ($m-$th MLDR) code. Also, if $d_m^H(D)$ denotes the $m-$ th GHW for code D, it is defined as

$$d_m^H(D) = min\{|supp(E)|; E \text{ is a } \mathbb{Z}_4 - \text{submodule of D and rank(E)} = m\},$$

The upper bound for $d_m^H(D)$ is $l - rank(D)$. The code meeting this upper bound is called MHDR code. In this paper, we investigate MLDR codes, MHDR codes, and relation between them in detail.

Keywords. Lee weight, Hamming weight, MHDR code, linear code, generalized Lee weight, MLDR code.

Mathematics Subject Classification (2020): $94B65$.

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Received 21 September 2025; Revised 05 November 2025; Accepted 22 December 2025
First Publish Date: 01 June 2026

1 Introduction and preliminaries

Finding an upper bound for computational concepts, has a long history in Coding Theory. One of them is Singleton bound which it is an upper bound for minimum Hamming weight of code. The code meets this bound is called MDS code, see [6]. Also, when Generalized Hamming Weight (GHW) introduced, the similar bound appeared entitled "Singleton type bound". We say that a code is $m-$th MDS code provided the $m-$th GHW of this code, meet this bound, see [3], [4] and [8]. In addition, we have the similar bounds for codes over rings and this issue provide MDR codes, complete information is in [1]. Now, if we substitute Lee weight and GLW rather than Hamming weight and GHW, an upper bound is obtained for GLW. The code meeting this bound is called MLDR code. Assume that $\mathbb{Z}_t$ is the alphabet of code. For any α in $\mathbb{Z}_t$, the Lee weight for α is denoted by $wt_L(\alpha)$ and defined as $wt_L(\alpha) = min\{t-\alpha, \alpha\}$. Substituting t with 4, namely in $\mathbb{Z}_4$, we gain the Lee weight for elements of $\mathbb{Z}_4$, namely

$$wt_L(\alpha) = \begin{cases} \alpha & \alpha \in \{0,2\} \\ 1 & \alpha \in \{1,3\}. \end{cases}$$

We define the Lee metric on $\mathbb{Z}_t^l$ as $wt_L(\alpha) = \Sigma_{j=1}^{l} wt_L(\alpha_j)$, where the sum is defined in N_0, the set of non-negative integers, and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)$. Also, Lee distance is defined as $wt_L(\alpha - \beta)$ and denoted by $d_L(\alpha, \beta)$. For more information, see [7]. Any subset of the free module $\mathbb{Z}_4^l$ is called a code of length l over $\mathbb{Z}_4$. Moreover, we say it is linear code of length l over $\mathbb{Z}_4$ if it is a $\mathbb{Z}_4-$ submodule of $\mathbb{Z}_4^l$, too. Consider D as a linear code over $\mathbb{Z}_4$ which has length l . Let $H(D)$ be an array which its rows are all code words of D. Note that $H(D)$ has $|D|$ rows and l columns. Assume s denotes any arbitrary column of $H(D)$. Then, s is corresponded to the following three cases:
a) s is composed of only zero's
b) s is composed of zero and two
c) s is composed of 0, 1, 2 and 3, namely all elements of $\mathbb{Z}_4$.
The Lee support weight for the column of the form (a) is defined as zero, it is defined 2 for column of the form (b) and 1 for (c). Moreover, the summation of all Lee support weights of the columns of $H(D)$ is defined as the Lee support of D which is denoted by $wt_L(D)$. Recall that the Lee weights of zero and two are themselves but the Lee weight of one and three are 1.

Example 1.1. *Let* $D = \{(0,0,0), (2,2,1), (2,3,1), (2,1,2), (0,1,2), (0,2,3), (0,3,3), (2,0,0)\}$. *Hence we have*

$$H(D) = \begin{pmatrix} 0\,0\,0 \\ 2\,2\,1 \\ 2\,3\,1 \\ 2\,1\,2 \\ 0\,1\,2 \\ 0\,2\,3 \\ 0\,3\,3 \\ 2\,0\,0 \end{pmatrix}.$$

Assume g_j be the j-th column of $H(D)$, then we have $wt_L(g_1) = 2$, $wt_L(g_2) = 1$ and $wt_L(g_3) = 1$. Hence we obtain that $wt_L(D) = 2 + 1 + 1 = 4$.

Assume D is a linear code over ring $\mathbb{Z}_4$. The $m-$ th generalized Lee weight with respect to rank, denoted by $d_m^L(D)$, is defined as follows

$$d_m^L(D) = min\{wt_L(E); E \text{ is a } \mathbb{Z}_4 - \text{submodule of D and rank(E)} = m\}.$$

Recall that for a linear code over ring $\mathbb{Z}_4$, the $m-$ th generalized Hamming weight with respect to rank, is defined as

$$d_m^H(D) = min\{|supp(E)|; E \text{ is a } \mathbb{Z}_4 - \text{submodule of D and rank(E)} = m\},$$

where $supp(E) = \{i : \exists v \in E with v_i \neq 0\}$.

In both definitions, we should have $1 \leq m \leq rank(D)$.

There is another generalization for Lee weight declared in [9].

2 Main results

Theorem 2.1. *[5] For code D of length l over finite chain ring A which has rank $k, 1 \leqslant m \leqslant k$, we have*

$$d_m^H(D) \leqslant l - k + r. \tag{1}$$

Definition 2.2. *The code meeting the upper bound in Theorem (2.1) is called Maximum Distance Separable with respect to Rank ($m-$ th MHDR) code.*

Remark 2.3. *It is easy to show that for any code, say D, and $m, 1 \leq m \leq rank(D)$, we have*

$$d_m^L(D) \leq 2d_m^H(D).$$

Proof. Assume $d_m^L(D) = wt_L(E)$ and $d_m^H(D) = |supp(E')|$, where E and E' are of rank m. By Definition of $d_m^L(D)$, we have

$$d_m^L(D) = wt_L(E) \leq wt_L(E'). \tag{2}$$

Consider $M(E')$ as in previous section. The column with zero in $M(E')$ will add value neither to $wt_L(E')$ nor to $|supp(E')|$, the column with 0, 2 equally will add 2 to $wt_L(E')$ and 1 to $|supp(E')|$ and the column with all elements of $\mathbb{Z}_4$ will add 1 to $wt_L(E')$ and $|supp(E')|$. So, we demonstrated that

$$wt_L(E') \leq 2|supp(E')|. \tag{3}$$

By using (2) and (3), we gain $d_m^L(D) \leq 2d_m^H(D)$.

□

The following theorem is a singleton type bound for GLW and we give an extended proof for it.

Theorem 2.4. *[2] Let D is a linear code of length l over $\mathbb{Z}_4$ which has rank k. Then for any $m, 1 \leqslant m \leqslant rank(D)$, we have*

$$\lfloor \frac{d_m^L(D) - 2m + 1}{2} \rfloor \leqslant l - k. \tag{4}$$

Proof. We write d_m^L instead of $d_m^L(D)$. Assume

$$\lfloor \frac{d_m^L(D) - 2m + 1}{2} \rfloor > l - k. \tag{5}$$

(i) If d_m^L is even, then $d_m^L - 2m$ is even. So, we can write $\lfloor \frac{d_m^L(D) - 2m + 1}{2} \rfloor = \lfloor \frac{d_m^L(D) - 2m}{2} + \frac{1}{2} \rfloor$ in which $\frac{d_m^L(D) - 2m}{2}$ is an integer number. Therefore, $\lfloor \frac{d_m^L(D) - 2m + 1}{2} \rfloor = \frac{d_m^L - 2m}{2}$. By substituting in (5), we have

$$\frac{d_m^L - 2m}{2} > l - k$$

and so

$$d_m^L > 2l - 2k + 2m. \tag{6}$$

Since $d_m^L(D) \leq 2d_m^H(D)$ and $d_m^H(D) \leq l - k + m$, we have

$$d_m^L \leqslant 2d_m^H \leqslant 2l - 2k + 2m$$

and thus

$$d_m^L \leqslant 2l - 2k + 2m. \tag{7}$$

Regarding (6) and (7), we have cotradiction.
(ii) If d_m^L is odd, then by similar method in (i), we obtain that d_m^L is even. It is a contradiction since d_m^L is odd. □

3 MLDR Codes

Definition 3.1. *Assume D is a linear code over $\mathbb{Z}_4$ which has length l. We say that D is an $m-$ th Maximum Lee Distance Separable with respect to Rank ($m-$ th MLDR) code whenever D meets the bound in inequality (4) in Theorem (2.4), namely, $\lfloor \frac{d_m^L(D) - 2m + 1}{2} \rfloor = l - k$.*

The following theorems are in [2] which give the important bound for GLW of codes and we write extended proof for them.

Theorem 3.2. *Assume D is an $m-$ th MLDR code, so we have $d_m^L(D) = 2d_m^H(D) - 1$ or $2d_m^H(D)$.*

Proof. At the first, by using Remark (2.3), we have

$$d_m^L(D) \leq 2d_m^H(D). \tag{8}$$

It is sufficient to prove that the inequality $d_m^L(D) < 2d_m^H(D) - 1$ does not hold. Let $d_m^L(D) < 2d_m^H(D) - 1$. Since D is MLDR code, the following obtained by Definition (3.1)

$$\lfloor \frac{d_m^L(D) - 2m + 1}{2} \rfloor = l - rank(D). \tag{9}$$

(i) If $d_m^L(D)$ is odd, then $d_m^L(D) + 1$ and $d_m^L(D) + 1 - 2m$ are even. Hence, $\frac{d_m^L(D) + 1 - 2m}{2}$ is an integer. So, $\lfloor \frac{d_m^L(D) + 1 - 2m}{2} \rfloor = \frac{d_m^L(D) + 1 - 2m}{2}$. Therefore, we can rewrite (9) as follows

$$\frac{d_m^L(D) + 1 - 2m}{2} = l - rank(D).$$

Thus

$$d_m^L(D) + 1 - 2m = 2l - 2rank(D),$$

and so

$$d_m^L(D) = 2l - 2rank(D) + 2m - 1.$$

By substituting the recent equality in $d_m^L(D) < 2d_m^H(D) - 1$, we obtain

$$2l - 2rank(D) + 2m - 1 < 2d_m^H(D) - 1.$$

Hence

$$2(l - rank(D) + m) < 2d_m^H(D),$$

which implies that

$$l - rank(D) + m < d_m^H(D).$$

It is contradiction according to Theorem 2.1.

(ii) If $d_m^L(D)$ is even, $\frac{d_m^L(D) - 2m}{2}$ is an integer and by the method similar to (i), we have

$$\lfloor \frac{d_m^L(D) + 1 - 2m}{2} \rfloor = \lfloor \frac{d_m^L(D) - 2m}{2} + \frac{1}{2} \rfloor = \lfloor \frac{d_m^L(D) - 2m}{2} \rfloor = \frac{d_m^L(D) - 2m}{2}.$$

Hence

$$\frac{d_m^L(D) - 2m}{2} = l - rank(D),$$

and so

$$d_m^L(D) = 2l - 2rank(D) + 2m.$$

By substituting the recent relation in $d_m^L(D) < 2d_m^H(D) - 1$, we obtain

$$2l - 2rank(D) + 2m < 2d_m^H(D) - 1.$$

This means that

$$2l - 2rank(D) + 2m < 2d_m^H(D),$$

which implies that

$$d_m^H(D) > l - rank(D) + m.$$

It is a contradiction, by Theorem 2.1. Therefore, $d_m^H(D) \geq 2d_m^H - 1$. This means that

$$2d_m^H - 1 \leq d_m^H(D) \leq 2d_m^H \tag{10}$$

by using (8). Hence, the proof is complete. □

Theorem 3.3. *Assume D is an $m-$ th MLDR linear code over $\mathbb{Z}_4$ which has length l. Then D is an $m-$ th MHDR code, too.*

Proof. Considering that D is $m-$th MLDR code, we have

$$\lfloor \frac{d_m^L(D) + 1 - 2m}{2} \rfloor = l - rank(D). \tag{11}$$

By Theorem (3.2), we have $d_m^L(D) = 2d_m^H(D) - 1$ or $2d_m^H(D)$. Now, we have the following cases
(i) If $d_m^L(D) = 2d_m^H(D) - 1$, we obtain

$$\begin{aligned} \frac{d_m^L(D) + 1 - 2m}{2} &= \frac{2d_m^H(D) - 1 - 2m + 1}{2} \\ &= \frac{2d_m^H(D) - 2m}{2} = d_m^H(D) - m. \end{aligned}$$

Also, we have

$$\lfloor \frac{d_m^L(D) + 1 - 2m}{2} \rfloor = d_m^H(D) - m. \tag{12}$$

(ii) If $d_m^L(D) = 2d_m^H(D)$, we have

$$\frac{d_m^L(D) + 1 - 2m}{2} = \frac{2d_m^H(D) - 2m + 1}{2} = d_m^H(D) - m + \frac{1}{2}.$$

Since $d_m^H(D) - m$ is an integer, we have

$$\lfloor \frac{d_m^L(D) + 1 - 2m}{2} \rfloor = \lfloor d_m^H(D) - m + \frac{1}{2} \rfloor = d_m^H(D) - m. \tag{13}$$

By using (11), (12) and (13), we obtain $d_m^H(D) - m = l - rank(D)$. Therefore, $l - rank(D) + m = d_m^H(D)$. This means that D is MHDR. □

4 Conclusion

In this paper, we discussed on MLDR and MHDR codes over $\mathbb{Z}_4$ and investigated the relation between them. Specially, we considered that any MLDR code over $\mathbb{Z}_4$ is an MHDR code.

Funding

This research received no external funding.

Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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Citation: F. Farhang Baftani, A walk on MLDR and MHDR codes, J. Disc. Math. Appl. 11(2) (2026) 123–129.

https://doi.org/10.22061/jdma.2025.12537.1167