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Research Paper

Cayley graphs and G-graphs of gyro-groups

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Abstract. The present paper investigates the structural properties of Cayley graphs and G-graphs associated with certain gyro-groups, providing rigorous proofs for several key characteristics. Additionally, it offers a comprehensive review of specific classes of gyro-groups, including gyrocommutative gyro-groups, dihedral gyro-groups, and dihedralized gyro-groups. Subsequently, the paper derives and establishes significant properties of the corresponding G-graphs. The study culminates in an examination of the symmetry properties exhibited by the Cayley graphs and G-graphs of selected gyro-groups, contributing to a deeper understanding of their algebraic and combinatorial structures.

Keywords. gyro-group, Cayley graph, G-graph.
Mathematics Subject Classification (2020): 05C25, 05C62, 06C65, 05C35.

1 Introduction

Gyro-group theory, introduced by Ungar [23], represents a significant extension of the study of Lorentz groups and provides a rigorous mathematical framework for understanding relativistic mechanics. By integrating Einstein's velocity addition law with the group struc-

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ture of Lorentz transformations, Ungar established a profound connection between Thomas precession and non-Euclidean geometry. He generalized Thomas precession into Thomas rotation, demonstrating its automorphism properties and introducing gyro-groups as a novel algebraic structure to encapsulate their non-associative nature. This reformulation bridges classical group theory with hyperbolic geometry, offering deep insights into both theoretical and applied domains.

Gyro-groups are deeply interconnected with various branches of science, including mathematical physics, abstract algebra, and loop theory. They provide an elegant formalism for modeling phenomena such as relativistic velocity composition and Thomas precession, underscoring their importance in modern mathematics and physics. Notably, gyro-groups generalize classical groups while retaining structural similarities, allowing many group-theoretic theorems to extend to gyro-groups [20–22].

Motivated by these parallels, we investigate the behavior of Cayley graphs and G-graphs within the context of gyro-groups. The concept of G-graphs, introduced by Bretto [7], is particularly significant due to its role in constructing semi-symmetric graphs. Our analysis reveals that certain special graphs, including fullerene graphs, can be represented as G-graphs of gyro-groups.

In addition, we provide examples of G-graphs associated with specific types of gyro-groups, including gyro-commutative and dihedral gyro-groups. Gyro-groups have been classified up to order 31 [2], with the smallest known gyro-group being of order 8. Focusing on this classification, we analyze the graphs of gyro-groups of order 8, laying the groundwork for future investigations into higher-order gyro-groups.

Our study begins with the simplest algebraic structure, the groupoid, which is a set equipped with a single binary operation satisfying only closure. For further details, the reader is encouraged to consult [16, 19–21].

Definition 1.1 ([10]). *A groupoid $(G, \oplus)$ is called a* gyro-group *if its binary operation satisfies the following axioms:*

(G1) There exists $0 \in G$ such that $0 \oplus a = a$ for all $a \in G$, where 0 is the identity element;

(G2) For any $a \in G$, there exists a unique $b \in G$ such that $b \oplus a = 0$; this element b is the two-sided inverse of a, denoted by $\ominus a$;

(G3) For any $a, b \in G$, there exists an automorphism $\mathrm{gyr}[a, b] : G \to G$ such that for all $c \in G$,

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c;$$

(G4) For any $a, b \in G$, $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$.

The study of group representations through graph-theoretical tools, such as Cayley graphs and G-graphs, has been a significant area of research. While Cayley graphs are widely used due to their regular structure, they have certain limitations, such as providing limited information about the underlying group and failing to establish a one-to-one correspondence between group isomorphism and graph isomorphism. In contrast, G-graphs offer a more refined representation by addressing these shortcomings.

This paper focuses on extending the concepts of Cayley graphs and G-graphs to gyro-groups, which generalize the notion of groups. The investigation builds on prior works, including studies by Bussaban et al. [10] on Cayley graphs of gyro-groups and the introduction of G-graphs for finite groups by Bretto et al. [5]. A key distinction is that while the connectivity of G-graphs for groups depends on the generating set spanning the group, this condition does not necessarily hold for gyro-groups. This paper demonstrates that the spanning property does not guarantee the connectedness of G-graphs in the context of gyro-groups.

The paper contributes to the field by constructing and analyzing the G-graph structures of specific finite gyro-groups. Furthermore, it explores properties such as connectivity and symmetry, offering insights into the conditions under which G-graphs of gyro-groups exhibit symmetric characteristics. These findings deepen the understanding of gyro-group representations and highlight the utility of G-graphs as a powerful alternative to traditional Cayley graphs.

Definition 1.2 ([8])**.** *Let (G,S) be a group with a generating set*

$$S=\{s_1,s_2,s_3,\dots,s_k\},\quad k\geq 1.$$

For any $s\in S$, consider the left action of the subgroup $H=\langle s\rangle$ on G. This induces a partition $G=\bigsqcup\langle s\rangle x$, where $x\in T_s$ and T_s is a right transversal of $\langle s\rangle$. The cardinality of $\langle s\rangle$ is $o(s)$, the order of s. Consider the cycles

$$(s)x=(x,sx,s^2x,\dots,s^{o(s)-1}x)$$

of the permutation $g_s\colon x\mapsto sx$ on G. We define a graph, denoted by

$$\Phi(G,S)=(V,E,\epsilon),$$

as follows:

The vertices of $\Phi(G,S)$ are the cycles of g_s for $s\in S$, i.e.,

$$V=\bigsqcup_{s\in S}V_s,\quad where\quad V_s=\{(s)x\mid x\in T_s\}.$$

For two distinct vertices $(s)x,(t)y\in V$, if

$$|\langle s\rangle x\cap\langle t\rangle y|=p,\quad p\geq 1,$$

then there is a p-edge between $(s)x$ and $(t)y$. The graph $\Phi(G;S)$ is called a graph from groups *or a* G-graph*.*

Interesting features of G-graphs are described in detail in [1,4,6,7,9].

Definition 1.3. *Let (G,S) be a gyro-group with a generating set*

$$S=\{s_1,s_2,s_3,\dots,s_k\},\quad k\geq 1.$$

For any $s \in S$, consider the left action of the subgyrogroup $H = \langle s \rangle$ on G. This induces a partition $G = \bigsqcup(\langle s \rangle \oplus x)$, where $x \in T_s$ and T_s is a right transversal of $\langle s \rangle$. The cardinality of $\langle s \rangle$ is $o(s)$, the order of s. Consider the cycles

$$(s) \oplus x = \left(x, s \oplus x, (s \oplus s) \oplus x, \ldots, \underbrace{(s \oplus \cdots \oplus s)}_{o(s)-1 \text{ times}} \oplus x \right)$$

of the permutation $g_s \colon x \mapsto s \oplus x$ on G. We define a graph, denoted by $\Phi(G,S)$, as follows:

I) The vertices of $\Phi(G,S)$ are the cycles of g_s for $s \in S$, i.e.,

$$V = \bigsqcup_{s \in S} V_s, \quad \text{where} \quad V_s = \{(s) \oplus x \mid x \in T_s\}.$$

II) For two distinct vertices $(s) \oplus x, (t) \oplus y \in V$, if

$$|(\langle s \rangle \oplus x) \cap (\langle t \rangle \oplus y)| = p, \quad p \geq 1,$$

then there is a p-edge between $(s) \oplus x$ and $(t) \oplus y$. The graph $\Phi(G,S)$ is called a graph from gyro-groups *or a G-**gyro-graph**.*

2 Main Results

In this section, we present the results obtained from the study of G-gyro-graphs. Several examples are then provided to facilitate a deeper understanding of the subject. These examples are carefully selected and systematically organized to enhance clarity and to lay the groundwork for further theoretical analysis.

Example 2.1. *Consider the gyro-group $G_8 = \{0,1,2,3,4,5,6,7\}$ and $S = \{1,2\}$ with the gyroautomorphism $A = (16)(25)$ and I which denotes the identity map. Also its gyro table is as follows.*

Table 1. The gyro table of G_8.

$\oplus$	0	1	2	3	4	5	6	7	gyro	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7	0	I	I	I	I	I	I	I	I
1	1	0	3	2	5	4	7	6	1	I	I	A	A	A	A	I	I
2	2	3	0	1	6	7	4	5	2	I	A	I	A	A	I	A	I
3	3	5	6	0	7	1	2	4	3	I	A	A	I	I	A	A	I
4	4	2	1	7	0	6	5	3	4	I	A	A	I	I	A	A	I
5	5	4	7	6	1	0	3	2	5	I	A	I	A	A	I	A	I
6	6	7	4	5	2	3	0	1	6	I	I	A	A	A	A	I	I
7	7	6	5	4	3	2	1	0	7	I	I	I	I	I	I	I	I

Obviously S is a generating set for G_8. We compute the 2-cycles of

$$V_r = \{(r) \oplus x; x \in G_8, r \in S\}.$$

Finally, the set of vertices is obtained as follows:

$V_1 = \{(0,1),(2,3),(4,5),(6,7)\}$,

$V_2 = \{(0,2),(1,3),(4,6),(5,7)\}$.

Based on the findings presented, it can be concluded that:

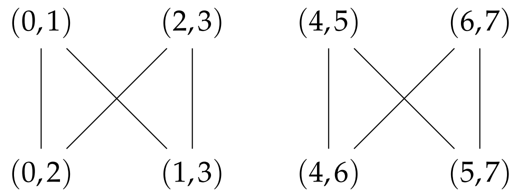

Figure 1. $\Phi(G_8, S = \{1,2\})$.

As you see in Figure 1, the G-graph of this gyro-group is disconnected with two isomorphic connected components of C_4. Note that although S is a generating set for G, $\Phi(G_8, S)$ is not connected. Just like the case for Cayley graph of gyro-groups.

Definition 2.2 ([10]). *Let S be a nonempty subset of a gyro-group G. The set S is said to be* symmetric *if for each element $s \in S$, we have $\ominus s \in S$. The* left-generating subset *of G by S, denoted by $(S\rangle$, is defined as*

$$(S\rangle = \{s_n \oplus (\cdots \oplus (s_3 \oplus (s_2 \oplus s_1))\cdots) \mid n \in \mathbb{N}, s_1, s_2, \cdots, s_n \in S\}.$$

If $(S\rangle = G$, we say that S left-generates *G, or that G is* left-generated *by S. The* right-generating *set is defined in a similar fashion [15].*

Therefore, we can state the following theorem.

Theorem 2.3. *Let G be a gyro-group and S is a non-empty subset of G. Then the G-graph $\Phi(G,S)$ is connected if and only if $(S\rangle = G$.*

Proof. First, let S be the left generating set of G of size one. We show that for every $x \in G$, there is a path from e to x and from x to e. Since $(S\rangle = G$, there exist $s_1, s_2, s_3, \cdots, s_n \in S$ such that

$$x = s_n \oplus (\cdots \oplus (s_3 \oplus (s_2 \oplus s_1))).$$

But we have assumed that S has only one element, which means

$x = s = s \oplus e = e \oplus s$. Thus the first part is proved.

Now suppose $\Phi(G,S)$ is connected. It is obvious that $(S\rangle \subseteq G$. Because of connectivity of $\Phi(G,S)$, there exists $s \in S$ such that $y = s \oplus e$, then $y \in (s\rangle$ and $G \subseteq (S\rangle$, which implies $(S\rangle = G$.

Now assume that $card(S) \geq 2$. For $s,s' \in S$, we show that there is a path from the set of vertices V_s to the set of vertices of $V_{s'}$. For $x,y \in G$ consider two vertices $(s) \oplus x \in V_s$ and $(s') \oplus y \in V_{s'}$. Since $G = (S\rangle$, there are $s_1,s_2,s_3,\cdots,s_n \in S$ such that

$$y = (s_n \oplus (\cdots \oplus (s_3 \oplus (s_2 \oplus s_1))\cdots)) \oplus x.$$

Moreover we have that:

$$\begin{aligned}
x &\in \langle s\rangle \oplus x \bigcap (s_1 \oplus x),\\
s_1 \oplus x &\in \langle s_1\rangle \oplus x \bigcap \langle s_0\rangle \oplus (s_1 \oplus x),\\
(s_2 \oplus s_1) \oplus x &\in \langle (s_2 \oplus s_1)\rangle \oplus x \bigcap \langle s_1\rangle \oplus ((s_2 \oplus s_1) \oplus x)),\\
(s_{n-1} \oplus \cdots \oplus (s_3 \oplus (s_2 \oplus s_1))) \oplus x) &\in \langle (s_{n-1} \oplus \cdots \oplus (s_3 \oplus (s_2 \oplus s_1)) \oplus x)\rangle\\
&\qquad \bigcap \langle s_n\rangle \oplus (s_{n-1} \oplus \cdots \oplus (s_3 \oplus (s_2 \oplus s_1)) \oplus x),\\
y &\in \langle (s_{n-1} \oplus \cdots \oplus (s_3 \oplus (s_2 \oplus s_1)) \oplus x\rangle \bigcap \langle s'\rangle \oplus y.
\end{aligned}$$

Therefore we could find a path from $(s) \oplus x \in V_s$ to $(s') \oplus y \in V_{s'}$. So $\Phi(G,S)$ is a connected graph.

Conversely, assume $\Phi(G,S)$ is connected and $x \in G$. There exists $s_{i_1} \in S$ and $x_{i_1} \in T_{s_{i_1}}$, such that $x \in (s_{i_1}) \oplus x_{i_1}$; hence $x = \underbrace{(s_{i_1} \oplus \cdots \oplus s_{i_1})}_{t_{i_1} times} \oplus x_{i_1}$.

We know that $e \in (s_{i_1})$ and the graph is connected if there exists a path from $x \in (s_{i_1}) \oplus x_{i_1}$ to $e \in ((s_{i_1})$; hence

$$\begin{aligned}
x &= \underbrace{(s_{i_1} \oplus \cdots \oplus s_{i_1})}_{t_{i_1} times} \oplus x_{i_1},\\
x_{i_1} &= \underbrace{(s_{i_2} \oplus \cdots \oplus s_{i_2})}_{t_{i_2} times} \oplus x_{i_2}, \cdots, x_{i_{k-2}} = \underbrace{(s_{i_{k-1}} \oplus \cdots \oplus s_{i_{k-1}})}_{t_{i_{k-1}} times} \oplus x_{i_{k-1}},\\
x_{i_{k-1}} &= \underbrace{(s_{i_k} \oplus \cdots \oplus s_{i_k})}_{t_{i_k} times} \oplus x_{i_k}.
\end{aligned}$$

But $x_{i_k} = e$, so $x = \underbrace{(s_{i_1} \oplus \cdots \oplus s_{i_1})}_{t_{i_1} times} \oplus \underbrace{(s_{i_2} \oplus \cdots \oplus s_{i_2})}_{t_{i_2} times} \oplus \cdots \oplus \underbrace{(s_{i_k} \oplus \cdots \oplus s_{i_k})}_{t_{i_k} times}$,

which shows that every element x of the gyro-group G can be written by the elements of the set S, this implies that G is left generated by S. □

In [10], Bussaban demonstrated that isomorphic gyro-groups do not necessarily lead to isomorphic corresponding Cayley graphs. In this section, the aforementioned issue is investigated within the framework of G-gyro-groups. To enhance the clarity of the problem, the discussion begins with the presentation of an example.

There are exactly six distinct gyro-groups of order 8, as identified in , with their respective gyro tables detailed in [2]. Notably, the gyro-groups $K(1)$, $L(1)$, $M(1)$, $N(1)$, and $O(1)$ introduced in [13] are isomorphic to those described in [2]. Among these is the fact that $G_{8,6} \cong N(1)$.

Example 2.4. *Consider two isomorphic gyro-groups $N(1)$ and $G_{8,6}$. We obtain the structure of $\Phi(G_{8,6}, S\{7,8\})$ and $\Phi(N(1), S = \{5,6,7\})$.*

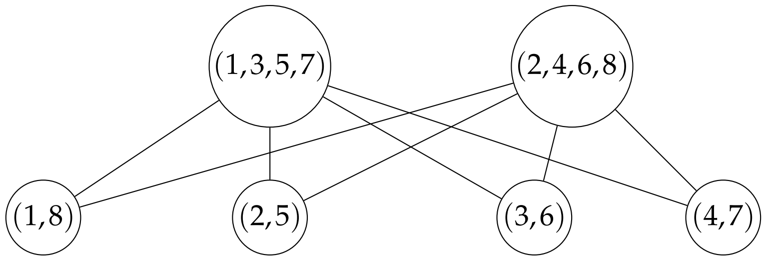

Figure 2. $\Phi(G_{8,6}\ , S = \{7,8\})$.

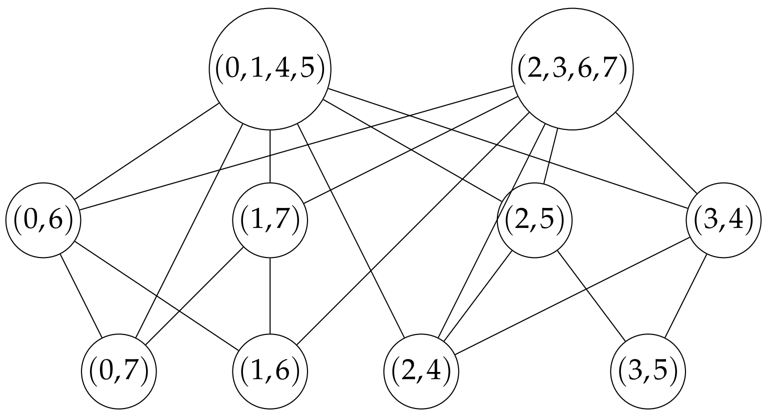

Figure 3. $\Phi(N(1), S = \{5,6,7\})$.

As you can see:

$$\Phi(G_{8,6}\ , \{7,8\}) \ncong \Phi(N(1), \{5,6,7\}).$$

This example shows that there are isomorphic gyro-groups with non-isomorphic G-graphs.

2.1 Examples of Special G-Gyro-graphs

In the following, we present examples of G-gyro-graphs associated with certain specific gyro-groups including gyro-commutative gyro-groups and dihedralized gyro-groups, along with complete definitions for each. In [12], G-graphs of 2-gyro-groups have been studied as a special class of gyro-groups.

Definition 2.5 ([21]). *(Gyro-commutative law) A gyro-group G is called a gyro-commutative gyro-group, if for all $a,b \in G$, we have:*

$$a \oplus b = gyr[a,b](b \oplus a).$$

Example 2.6. *Consider two isomorphic gyro-commutative gyro-groups $M(1)$ and $G_{8,5}$. Their G-gyro-graphs are:*

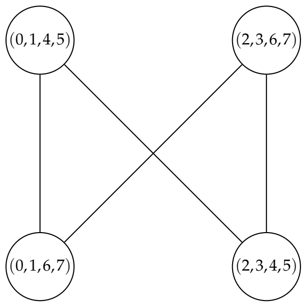

Figure 4. $\Phi(M(1), S = \{5,7\})$.

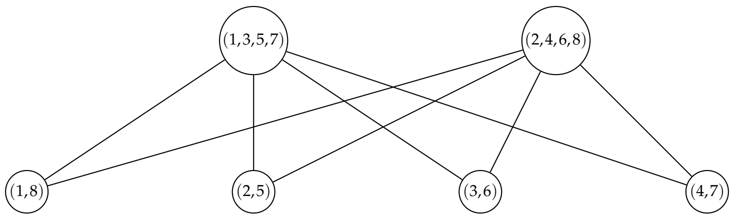

Figure 5. $\Phi(G_{8,5}, S = \{7,8\})$.

As you see that the G-graphs of these two isomorphic gyro-commutative gyro-groups are not isomorphic to each other.

Example 2.7. *Consider the gyro-group $Dih(G_8)$, for which the corresponding the gyro table has been presented in Example 5 ([17]).*

The set of vertices of the generalized dihedral gyro-group $Dih(G_8)$ is $V_{(4,0)}$ members with $0,1,2,3$, $V_{(0,1)}$ members with $4,5,\cdots,11$, and $V_{(3,0)}$ members with $12,13,\cdots,19$.

2.2 The Symmetry of Cayley graphs and G-graphs of gyro-groups

One of the fundamental concepts in graph theory is the study of automorphism groups of graphs. For example, in [1], the automorphism groups of Involution G-graphs and Cayley graphs are explored. This section examines the symmetry properties of Cayley graphs and G-graphs in the context of gyro-groups. A graph Λ is vertex-transitive if, for any $x,y \in V(\Lambda)$, there exists $f \in Aut(\Lambda)$ such that $f(x) = y$. Similarly, edge-transitivity is defined for

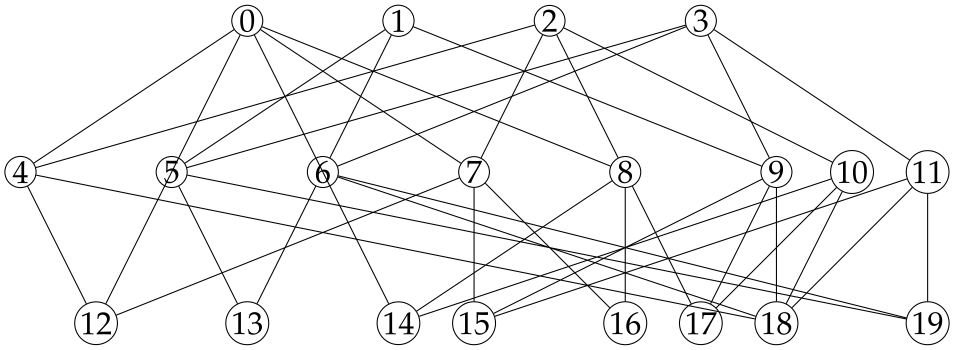

Figure 6. $\Phi(Dih(G_8), S = \{(3,0),(4,0),(0,1)\})$.

edges, and a graph possessing both properties is termed symmetric. In earlier sections, we analyzed the Cayley graphs of gyro-groups, noting that certain group-theoretic properties do not carry over to gyro-groups. Below, we investigate another distinguishing feature of these structures. Bussaban et al. in [10], demonstrated that, in contrast to Cayley graphs associated with groups, the Cayley graphs corresponding to gyro-groups are not necessarily transitive. To elucidate this distinction more clearly, an illustrative example is presented below.

Example 2.8 ([10]). *The Cayley graph of gyro-group* $(G, \oplus)$ *is defined in Example 2.1 with the generating set* $S = \{1,2,3\}$ *is not vertex-transitive.*

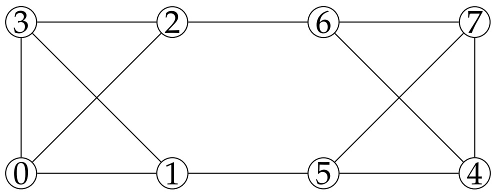

Figure 7. $Cay(G, \{1,2,3\})$.

In [15, 16] a sufficient condition is given for an L-Cayley graph and a R-Cayley graph to be transitive. In what follows, we introduce the necessary condition for the transitivity of G-gyro-graphs by presenting new definitions and notations.

Definition 2.9. *Let* G *be a gyro-group,* S *a subset of* G *so that* $S \neq \emptyset$. *If* S *be left generating set, i.e* $\langle S \rangle = G$, *we call the* G*-graph of* (G,S) *as*
L-G-graph (G,S). *If* S *is the right generating set, we call it R-G-graph* (G,S).

The subsequent example is provided to enhance the understanding of the subject.

Example 2.10. *Consider the gyro-group* G_{16}. *let* $S = \{1,2,3\}$ *be left generating set for* G_{16}. *It is clear that* S *is symmetric. Then we choose three arbitrary members* $5,6 \in G_{16}$ *and* $2 \in S$. *We have:*

$$gyr[5,2](6) = \ominus(5 \oplus 2) \oplus (5 \oplus (2 \oplus 6)) = 6.$$

On the other hand $gyr[g,s]$ is the identity map for all $g \in G$, $s \in S$. Then L-G-graph (G,S) is vertex-transitive.

Table 2. The addition table of the gyro-group G_{16}.

$\oplus$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	1	0	6	7	5	4	11	10	8	9	15	14	12	13
3	3	2	0	1	7	6	4	5	10	11	9	8	14	15	13	12
4	4	5	6	7	3	2	0	1	15	14	12	13	9	8	11	10
5	5	4	7	6	2	3	1	0	14	15	13	12	8	9	10	11
6	6	7	5	4	0	1	2	3	13	12	15	14	10	11	9	8
7	7	6	4	5	1	0	3	2	12	13	14	15	11	10	8	9
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	9	8	14	15	13	12	3	2	0	1	7	6	4	5
11	11	10	8	9	15	14	12	13	2	3	1	0	6	7	5	4
12	12	13	14	15	11	10	8	9	6	7	5	4	0	1	2	3
13	13	12	15	14	10	11	9	8	7	6	4	5	1	0	3	2
14	14	15	13	12	8	9	10	11	4	5	6	7	3	2	0	1
15	15	14	12	13	9	8	11	10	5	4	7	6	2	3	1	0

Theorem 2.11. *Let G be a finite gyro-group and S be a symmetric subset of G. If $gyr[g,s]$ is the identity map for all $g \in G$, $s \in S$, then L-G-graph (G,S) is vertex-transitive.*

Proof. In fact, we prove that the condition on $gyr[g,s]$ causes right addition by any $g \in G$, is automorphism on L-G-graph (G,S). Let $(u) \oplus x$ and $(v) \oplus y$ be two vertices in L-G-graph (G,S), then

$$(\langle u \rangle \oplus x \bigcap \langle v \rangle \oplus x) = p \geq 1.$$

Now, Let $\langle w \rangle \oplus x$ and $\langle z \rangle \oplus x$ is adjacent in L-G-graph (G,S). Therefore

$$(\langle w \rangle \oplus x \bigcap \langle z \rangle \oplus y) = m,$$

where m can be a single point or a cyclic. We add g to both sides on the right. Then we have:

$$\begin{aligned}
&(\langle w \rangle \oplus x \bigcap \langle z \rangle \oplus y) \oplus g = l \\
&= (((\langle w \rangle \oplus x) \oplus g) \bigcap (((\langle z \rangle \oplus y) \oplus g) = l \\
&= ((x, w_1 \oplus x, \cdots, w_n \oplus x) \oplus g) \bigcap ((y, z_1 \oplus x, \cdots, z_k \oplus y) \oplus g) = l \\
&= ((x \oplus g, (w_1 \oplus x) \oplus g, \cdots, (w_n \oplus x) \oplus g)) \\
&\cap ((y \oplus g, (z_1 \oplus y) \oplus g, \cdots, (z_k \oplus y) \oplus g)) \\
&= ((x \oplus g, w_1 \oplus (x \oplus gyr[w_1, x]), \cdots, w_n \oplus (x \oplus gyr[w_n, x]) \\
&\cap ((y \oplus g, z_1 \oplus (y \oplus gyr[z_1, y]), \cdots, z_k \oplus (y \oplus gyr[z_k, y]) = l.
\end{aligned}$$

Since $gyr[g,s]$ is the identity map for all $g \in G$, $s \in S$ then:

$$\begin{aligned}
&((x \oplus g, w_1 \oplus (x \oplus g), \cdots, w_n \oplus (x \oplus g)) \\
&\bigcap ((y \oplus g, z_1 \oplus (y \oplus g), \cdots, z_k \oplus (y \oplus g)) \\
&= ((w_1, \cdots, w_n) \oplus (x \oplus g)) \bigcap ((z_1, \cdots, z_k) \oplus (y \oplus g)) \\
&= ((w) \oplus (x \oplus g)) \bigcap ((z) \oplus (y \oplus g)) = l.
\end{aligned}$$

Since $((w) \oplus (x \oplus g))$ and $((z) \oplus (y \oplus g))$ are adjacent, the mapping $g_s = G \longmapsto G$ defined by $x \mapsto s \oplus x$, is an automorphism that sends $(u) \oplus x$ to $(v) \oplus y$. Therefore L-G-graph is vertex-transitive. □

The same result holds for R-G- graphs.

Theorem 2.12. *Let G be a finite gyro-group with a symmetric non-empty subset S. Assume that $gyr[g,g'](S) = S$ is the identity map for all $g, g' \in G$, then R-G-graph (G,S) is vertex-transitive.*

Proof. Proof is similar to previous theorem. □

Example 2.13. *Consider $(G_{16}, S = \{8,9\})$, where S is symmetric, $(S\rangle = G$ and $gyr[g,g'](S) = S$ for $g, g' \in G$. We see R-G-graph (G,S) is vertex-transitive.*

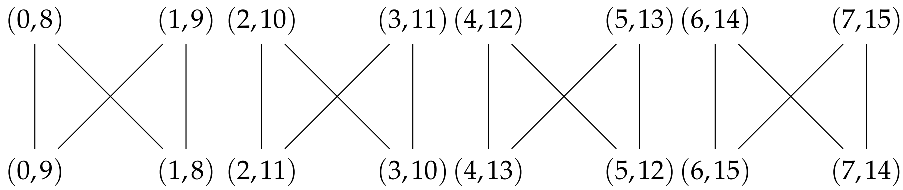

Figure 8. Cayley graph of G_{16}.

The following definition is presented as a necessary foundation for proving the subsequent theorems and propositions.

Definition 2.14. *[7] The line graph $L(\Gamma)$ associated with a simple graph Γ has for vertices the edges of Γ, two vertices being adjacent if and only if the corresponding edges in Γ are adjacent.*

Proposition 2.15. *Let G be a gyro-group and $S = \{s,t\}$ with $(S\rangle = G$ and $\langle s \rangle \cap \langle t \rangle = \{0\}$. Then Γ= L-G-graph (G,S) is a simple graph and $L(\Gamma) \simeq Cay(G,A)$ where $A = (\langle s \rangle \cup \langle t \rangle) \setminus \{0\}$.*

Proof. Let $\theta : G \longrightarrow V(L(\Gamma))$ be the map that assigns to each $u \in G$ the vertex in the line graph $L(\Gamma)$ corresponding to the edge $a = \big([(s) \oplus x, (t) \oplus y], u\big) \in E(\Gamma)$, where $\langle s \rangle \oplus x \cap \langle t \rangle \oplus y = \{u\}$. Since Γ is constructed as a G-graph, the generating property ensures that such an edge exists for every $u \in G$. Hence, θ is well-defined.

Now suppose $u = u'$. Then the edge associated with u is the same as that associated with u', so their images under θ coincide: $\theta(u) = \theta(u')$. This confirms that θ is well-defined.

To prove injectivity, assume $\theta(u) = \theta(u')$. Then the corresponding edges are equal:

$$\big([(s) \oplus x, (t) \oplus y], u\big) = \big([(s) \oplus x', (t) \oplus y'], u'\big).$$

By construction, the label of the edge determines the element of G, so $u = u'$. Thus, θ is injective.

For surjectivity, note that every vertex in $V(L(\Gamma))$ corresponds to an edge in Γ, and each such edge is labeled by some $u \in G$, due to the unique intersection of left cosets. Since every such label arises from some $u \in G$, the image of θ covers all vertices of $L(\Gamma)$.

Therefore, θ is a well-defined bijection.

□

Example 2.16. *Consider the gyro-group G_8 with the generating set $S = \{1,3\}$. The corresponding G-graph of this group is as follows.*

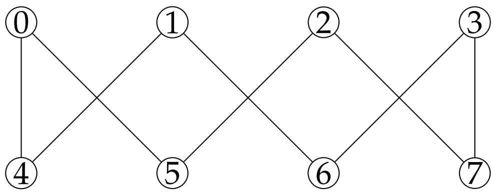

Figure 9. $\Phi(G_8, \{1,3\})$.

Example 2.17. *Consider $G_8, S = \{1,3\}$, such that each edge $n - m$ is denoted by a letter as follows, then*

$$04 = a, 05 = e, 16 = g, 27 = p, 14 = f, 25 = l, 36 = q, 37 = r,$$

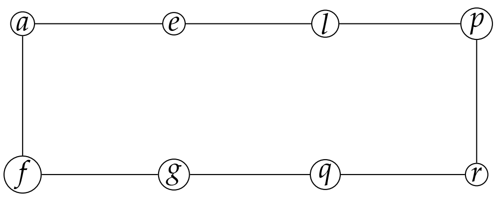

Figure 10. $L(\Phi(G_8, \{1,3\}))$.

Finally, we drew the Cayley graph $(G_8, A = (\langle 1 \rangle \cup \langle 3 \rangle) \setminus \{0\})$

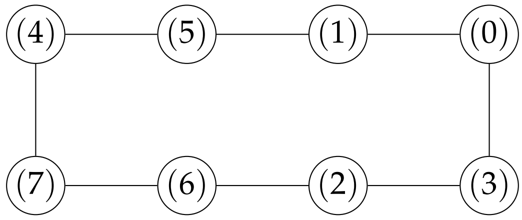

Figure 11. $Cay(G_8, A = (\{\langle 1\rangle \cup \langle 3\rangle) \setminus \{0\})$.

As we see, the line graph (G-graph $(G_8, \{1,3\})$*) is isomorphic with Cayley graph*

$$(G_8, A = (\langle 1\rangle \cup \langle 3\rangle) \setminus \{0\}))\,.$$

Proposition 2.18 ([18]). *A connected graph is edge-transitive if and only if the line graph of its vertices is transitive.*

After the preliminaries, we will come to the most important point of this section.

Theorem 2.19. *Let G be a gyro-group and* $S = \{s, t\}$ *with* $(S\rangle = G$ *and*
$\langle s\rangle \cap \langle t\rangle = \{0\}$ *and also* $gyr[s, g]$ *is identity map for* $g \in G$ *and* $s \in S$*, then L-G-graph* (G, S) *is symmetric.*

Proof. According to theorem 2.11, this graph is vertex-transitive. According to proposition 2.15 line graph of this graph is isomorphic with Cayley graph and is vertex-transitive. By using of the proposition 2.18, we can see this graph is edge-transitive. This completes the proof. □

Remark 2.20. *According to Bussaban's article, the graph of gyro-groups is not a vertex-transitive graph. But it can be a edge-transitive graph, and so it can be semi-regular. This is despite the fact that cayley graph of groups are always transitive vertices. And they can never be semi-regular.*

3 Identifying G-graphs of Gyro-groups

This section investigates the problem of determining whether a given G-graph corresponds to a gyro-group graph. To identify the underlying gyro-group G and a generating set S such that $\Phi(G, S)$ is isomorphic to a given graph Γ, the following methodology is proposed:

- For graphs of order less than 32, the classification provided in [2] can be utilized, as it lists all gyro-groups of small orders up to isomorphism.
- For graphs of order 32 or greater, a complete classification of gyro-groups is currently unavailable, and such algebraic structures have not yet been implemented in computational algebra systems such as GAP. Nonetheless, this limitation is expected to be addressed in future research.

To illustrate the approach, we demonstrate that certain well-known graphs are G-gyro-graphs by identifying their corresponding gyro-groups. Consider the **Heawood graph**, a bipartite graph with 14 vertices and 21 edges, in which each vertex has degree 3. Assuming that this graph can be realized as a G-graph $\Phi(G,S)$, the associated gyro-group G must have order 21 and be generated by a set S of size 2, as implied by the graph's structure. According to the classification of gyro-groups of small orders, there exists a unique gyro-group of order 21. Constructing $\Phi(G_{21},S)$ yields a graph isomorphic to the Heawood graph, thereby confirming that the Heawood graph is indeed a G-gyro-graph.

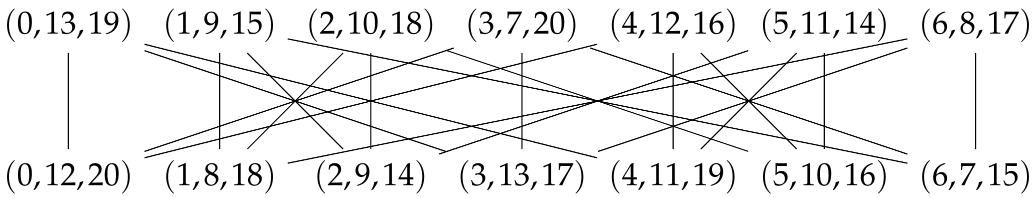

Figure 12. $\Phi(G_{21},S=\{19,20\})$.

4 Conclusion

In this paper, we have investigated the structural and symmetry properties of Cayley graphs and G-graphs associated with gyro-groups an important class of non-associative algebraic structures that generalize groups and arise naturally in hyperbolic geometry and relativistic physics. Our results demonstrate that, unlike their group-theoretic counterparts, Cayley graphs and G-graphs of gyro-groups need not be connected or vertex-transitive, even when the generating set left-generates the entire gyro-group. This highlights a fundamental departure from classical group-based graph constructions.

Through explicit examples involving small-order gyro-groups particularly of orders 8 and 16 we have shown that isomorphic gyro-groups may yield non-isomorphic G-graphs. This underscores the sensitivity of G-graph representations to both the choice of generating set and the internal gyration structure, indicating that such graphical models encode more refined combinatorial information than algebraic isomorphism alone.

We established sufficient conditions for vertex- and edge-transitivity in G-graphs of gyro-groups. In particular, we proved that if all gyroautomorphisms induced by elements of the generating set act trivially (i.e., as the identity map), then the corresponding L-G-graph is vertex-transitive. Furthermore, by leveraging the relationship between G-graphs and their line graphs, we linked the symmetry of these graphs to the vertex-transitivity of associated Cayley graphs, thereby deriving conditions under which G-graphs become symmetric (i.e., both vertex- and edge-transitive).

Finally, using the classification of gyro-groups of small order, we demonstrated that well-known combinatorial objectssuch as the Heawood graph can be realized as G-graphs of specific finite gyro-groups. This not only enriches the landscape of algebraic graph theory but also positions gyro-groups as a viable framework for modeling symmetric and semi-symmetric graphs beyond the scope of classical group actions.

Collectively, our findings advance the interplay between non-associative algebra and graph theory, revealing both the expressive power and the nuanced behavior of gyro-group based graph constructions. They also open promising avenues for future research, including the classification of symmetric G-gyro-graphs, spectral analysis of such graphs, and potential applications in areas where non-associativity plays a fundamental role.

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Data Availability Statement

No data is used by the authors in this article.

Conflicts of Interests

The authors declare that there is no conflict of interest.

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